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Extended Rayleigh Distribution: Properties and Application to Failure Data

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Authors' contributions

This work was carried out in collaboration among all authors. All authors read and approved the final manuscript.

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Abstract

This paper introduces a new three parameter Rayleigh distribution which generalizes the Rayleigh distribution. The new model is referred to as Extended Rayleigh (ER) distribution. Various mathematical properties of the new model including ordinary and incomplete moment, quantile function, generating function are derived. We propose the method of maximum likelihood for estimating the model parameters. A real life data set is used to compare the flexibility of the new model with other models.

Keywords: Extended Rayleigh distribution; maximum likelihood; incomplete moment; order statistics.

1 Introduction

In several areas such as survival analysis and other applied area of statistics, there is strong need to develop more flexible classical distributions. Based on this assertion, different methods of generating new families of distributions have been established. This includes beta-G by Eugene et al. (2002), skew family by Azzalini

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(1985), Cordeiro and de Castro (2011) studied Kumaraswamy-G family of distribution, Kumaraswamy transmuted-G by Afify et alo (2016), the odd Lindley-G family of distributions by Silva, Percontini, de Brito, Ramos, Venancio and Cordeiro [1], exponentiated transmuted-G family by Merovci, Alizadeh, Yousof and Hamedani [2], the odd-Burr generalized family by Alizadeh, Cordeiro, Nascimento, Lima and Ortega [3], the transmuted Weibull-G family by Alizadeh, Rasekhi, Yousof and Hamedani [4], the type I half logistic family by Cordeiro, Alizadeh and Diniz Marinho [5], the Zografos–Balakrishnan odd log-logistic family of distributions by Cordeiro, Alizadeh, Ortega and Serrano [6], logistic-X by Tahir, Cordeiro, Alzaatreh, Mansoor and Zubair [7], a new Weibull-G by Tahir Mansoor, Cordeiro, Alizadeh and Hamedani [8] and many more. According to Kharazmi and Saadatinik (2018), the hyperbolic Sine (HS) family with cumulative density function given by

$$F(x) = \frac{2e^{\rho}}{(e^{\rho}-1)^2}(cosh\{\rho G(x)\} - 1) \tag{1}$$

And the probability density function (pdf) is

$$f(x) = \frac{2\rho e^{\rho}}{(e^{\rho}-1)^2} g(x) sinh\{\rho G(x)\} \tag{2}$$

Where, $G(x)$ and $g(x)$ are the cdf and pdf for any random variable, respectively and the hyperbolic sine function sinh (x) is defined as

$$Sinh(x) = {}^{1}\!/_{2}\,(e^{x} - e^{-x}) \tag{3}$$

and using the series expansion $Sinh(x)$ takes the following form

$$Sinh(x) = \sum_{j=0}^{\infty} \frac{x^{2j+1}}{(2j+1)!} \tag{4}$$

In our work, we take $g(x)$ is the CDF of the Rayleigh distributiuon and $g(x)$ its pdf. The cdf of Rayleigh distribution is given by

$$G(x) = 1 - e^{(-\eta x)^2}, \qquad \text{x} > 0 \tag{5}$$

Where η is a positive shape parameter representing the characteristics of the distribution.

The associated pdf is

$$g(x) = 2\eta^2 x e^{(-\eta x)^2} \tag{6}$$

2 The New Model

This section contributes the representation of ER distribution. The cdf, reliability, hazard rate, cumulative hazard rate functions are obtained and discussed analytically. As well as the asymptotic behavior of ER distribution.

2.1 Mathematical representations

By putting equation (5) and (6) into (1) and (2), then the ER cdf and pdf will be obtained as

$$J(x) = \frac{2e^{\rho}}{(e^{\rho}-1)^2}\left(cosh\left\{1 - e^{(-\eta x)^2}\right\} - 1\right) \tag{7}$$

and

$$j(x) = \frac{4\rho e^{\rho}}{(e^{\rho}-1)^2}\eta^2 x e^{(-\eta x)^2} sinh\left(\rho\left(1-e^{(-\eta x)^2}\right)\right) \tag{8}$$

The survival and the hazard rate functions are

$$S(x) = 1 - \frac{2e^{\rho}}{(e^{\rho}-1)^2}\left(cosh\{1-e^{(-\eta x)^2}\}-1\right) \tag{9}$$

$$h(x) = \frac{\frac{4\rho e^{\rho}}{(e^{\rho}-1)^2}\eta^2 x e^{(-\eta x)^2} sinh\left(\rho\left(1-e^{(-\eta x)^2}\right)\right)}{1-\frac{2e^{\rho}}{(e^{\rho}-1)^2}\left(cosh\rho\{1-e^{(-\eta x)^2}\}-1\right)} \tag{10}$$

in addition, the cumulative hazard rate function corresponding to (10) is

$$H(x) = -lns(x) = -ln\left\{1-\frac{2e^{\rho}}{(e^{\rho}-1)^2}\left(cosh\{1-e^{(-\eta x)^2}\}-1\right)\right\} \tag{11}$$

The graph of the pdf of ER distribution is given in Figs. 1 and 2 as drawn below

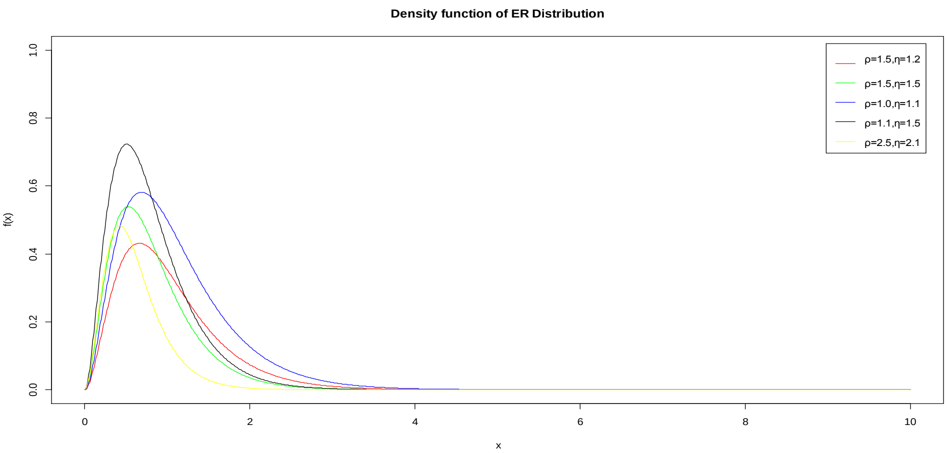

Fig. 1. Graph of the density function of Extended Rayleigh distribution

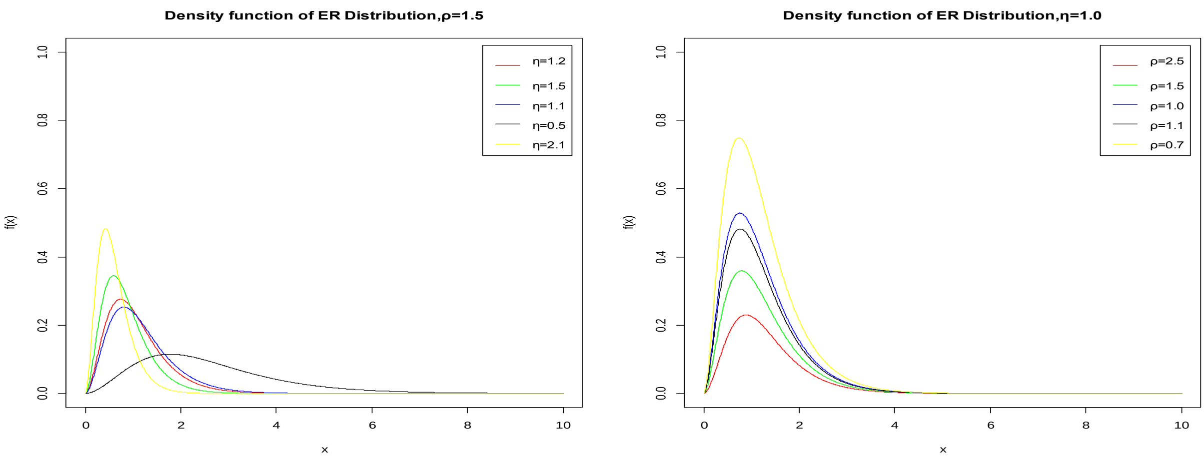

Fig. 2. Graph of the density function of Extended Rayleigh distribution

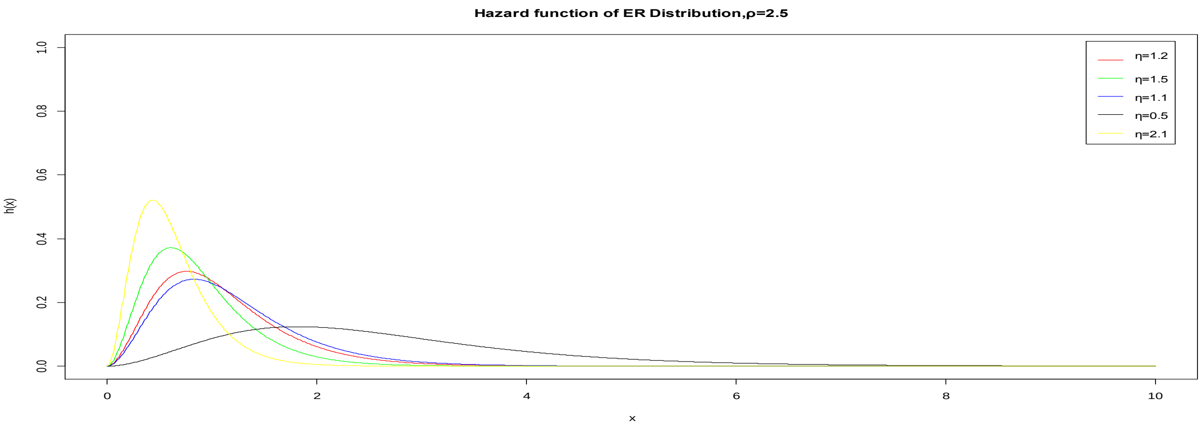

Fig. 3. Graph of the hazard function of Extended Rayleigh distribution

This implies that using Taylor series given in (3) and (4), the pdf of ER distribution can be written as

$$j(x) = \frac{4\rho e^{\rho}}{(e^{\rho}-1)^2}\eta^2 \sum_{j=0}^{\infty}\sum_{k=0}^{2j+1}\frac{\rho^{2j+1}}{(2j+1)!}(-1)^k\binom{2j+1}{k}xe^{-\eta^2 x^2(k+1)} \quad (12)$$

2.2 Quantile function

The quantile function (qf) say x_q, of X is the real solution of the following equation $F(x_q) = q$. Then we can write

$$q = \frac{2e^{\rho}}{(e^{\rho}-1)^2}\left(cosh\rho\left\{1 - e^{(-\eta x)^2}\right\} - 1\right) \quad (13)$$

By making x the subject of the formula we obtain the u^{th} quantile for the ER distribution as

$$x_q = \left(-\frac{1}{\eta^2}\left[ln\left\{1 - cosh^{-1}\left(\frac{1}{2}exp(-\rho)(q+1)(exp(\rho)-1)^2\right)\right\}\right]\right)^{1/2} \quad (14)$$

By putting $q = 0.5$ in equation (14) gives the median of X. Simulating the ER distribution is straightforward.

If U is uniform variate on the unit interval $(0,1)$, then the random variable $X = X_u$ at $q = U$ follows (4).

3 Moments and Incomplete Moment

3.1 r^{th} raw moment

Moment of a distribution plays a very important role in statistical analysis. They are used for estimating characteristics of a distribution such as kurtosis, skewness and measures of central tendency and measures of dispersion. The r^{th} moment, denoted by μ_r, of X is given by

$$E(x^r) = \mu_r = \int_{-\infty}^{\infty} x^r f(x)dx \quad (15)$$

Putting equation (12) in (15)

$$E(x^r) = \frac{4\rho e^{\rho}}{(e^{\rho}-1)^2}\eta^2 \sum_{j=0}^{\infty}\sum_{k=0}^{2j+1} \frac{\rho^{2j+1}}{(2j+1)!}(-1)^k \binom{2j+1}{k} \int_{\mathbf{0}}^{\infty} \boldsymbol{x^{r+1}} e^{-\eta^2 x^2 (k+1)} dx \tag{16}$$

Taking $m = \eta^2 x^2 (k+1),\ x = \frac{m^{1/2}}{\eta(k+1)^{1/2}}, dx = \frac{1/2 m^{-1/2}}{\eta(k+1)^{1/2}}$ and substitute in equation (16), we have the r^{th} moment of Extended Rayleigh distribution given by

$$E(x^r) = \frac{2\rho e^{\rho}}{(e^{\rho}-1)^2} \sum_{j=0}^{\infty}\sum_{k=0}^{2j+1} \frac{\rho^{2j+1}}{(2j+1)!}(-1)^k \binom{2j+1}{k} (k+1)^{-(r/2+1)\eta^r \Gamma(r/2+1)} \tag{17}$$

3.2 Incomplete moment

The incomplete moment is used to estimate the median deviation, mean deviation and measures of inequalities such as the Lorenz and Bonferroni curves. The incomplete moment of a distribution is given by

$$\varphi(t) = \int_t^{\infty} x^r f(x) dx \tag{18}$$

Putting equation (12) in (18), then we have

$$\varphi(t) = \frac{4\rho e^{\rho}}{(e^{\rho}-1)^2}\eta^2 \sum_{j=0}^{\infty}\sum_{k=0}^{2j+1} \frac{\rho^{2j+1}}{(2j+1)!}(-1)^k \binom{2j+1}{k} \int_{\boldsymbol{t}}^{\infty} \boldsymbol{x^{r+1}} e^{-\eta^2 x^2 (k+1)} dx \tag{19}$$

Taking $m = \eta^2 x^2 (k+1),\ x = \frac{m^{1/2}}{\eta(k+1)^{1/2}}, dx = \frac{1/2 m^{-1/2}}{\eta(k+1)^{1/2}}$ and substitute in equation (19), we have the r^{th} incomplete moment of

$$\varphi(x) = \frac{2\rho e^{\rho}}{(e^{\rho}-1)^2} \sum_{j=0}^{\infty}\sum_{k=0}^{2j+1} \frac{\rho^{2j+1}}{(2j+1)!}(-1)^k \binom{2j+1}{k} (k+1)^{-(r/2+1)\eta^r \Gamma(r/2+1, \eta^2 t^2 (k+1)} \tag{20}$$

3.3 Moment generating functions

Moment generating function is a very useful function that can be used to describe certain properties of the distribution. It can be used to obtain moments of a distribution. The moment generating function of ER distribution is obtained as follows:

the moment generating function of a random variable X is given by

$$M_X(t) = E(e^{tX}) = \int_{-\infty}^{\infty} e^{tX} f(x) dx \tag{21}$$

Where f(x) is given in (12). Using series expansion for e^{tX} given by

$$e^{tX} = \sum_{r=0}^{\infty} \frac{t^r}{r!} x^r \tag{22}$$

Using (22), we can re-write equation (21) as follows

$$M_X(t) = \int_{-\infty}^{\infty} \sum_{r=0}^{\infty} \frac{t^r}{r!} x^r f(x) dt = \sum_{r=0}^{\infty} \frac{t^r}{r!} E(X^r) \tag{23}$$

Putting equation (17) in (23), we have an expression for the moment generating function of ER distribution as

$$M_X(t) = \sum_{j,r=0}^{\infty} \sum_{k=0}^{2j+1} \frac{t^r}{r!} \frac{2\rho e^\rho}{(e^\rho - 1)^2} \frac{\rho^{2j+1}}{(2j+1)!} (-1)^k \binom{2j+1}{k} (k+1)^{-(r/_2+1)\eta^r \Gamma(r/_2+1)} \tag{24}$$

4 Estimation of the Parameters of ER Distribution

The likelihood function of ER distribution is given by

$$L(\Omega) = \prod_{i=1}^{n} \left\{ \frac{4\rho e^\rho}{(e^\rho - 1)^2} \eta^2 x e^{(-\eta x)^2} sinh\left(\rho\left(1 - e^{(-\eta x)^2}\right)\right) \right\} \tag{25}$$

The log-likelihood function $l(\Omega) = log(L(\Omega))$ of the ECTE distribution is given by

$$l = nlog\left(\frac{4\rho e^\rho}{(e^\rho - 1)^2} \eta^2\right) + \sum_{i=1}^{n} \log\ (x) - \eta^2 \sum_{i=1}^{n} x^2 + \sum_{i=1}^{n} x^2 sinh\left(\rho\left(1 - e^{(-\eta x)^2}\right)\right) \tag{26}$$

4.1 Application

The Extended Rayleigh distribution is used to model a life time data and compared with other competing distributions such as the Exponential distribution (E), Gumbel type-2 (GT) distribution and the Exponentiated Gumbel type-2 (EGT) distribution. The data represents the survival times (in days) of 72 guinea pigs infected with virulent tubercle bacilli, observed and reported by Bjerkedal et al. [9]. The data obtained is given as:

0.1, 0.33, 0.44, 0.56, 0.59, 0.72, 0.74, 0.77, 0.92, 0.93, 0.96, 1, 1, 1.02, 1.05, 1.07, 1.07, 1.08, 1.08, 1.08, 1.09, 1.12, 1.13, 1.15, 1.16, 1.2, 1.21, 1.22, 1.22, 1.24, 1.3, 1.34, 1.36, 1.39, 1.44, 1.46, 1.53, 1.59, 1.6, 1.63, 1.63, 1.68, 1.71, 1.72, 1.76, 1.83, 1.95, 1.96, 1.97, 2.02, 2.13, 2.15, 2.16, 2.22, 2.3, 2.31, 2.4, 2.45, 2.51, 2.53, 2.54, 2.54, 2.78, 2.93, 3.27, 3.42, 3.47, 3.61, 4.02, 4.32, 4.58, 5.55.

Exploratory Data Analysis (EDA) is given in Table 1 which clearly shows that the data is positively skewed and under-dispersed. Estimates of the parameters of the Extended Rayleigh and Rayleigh distribution, Akaike information criterion (AIC), consistent Akaike information criterion (CAIC), Bayesian information criterion (BIC), Crammer Von-misses (W) and Anderson Darling (A*) statistic are also given in Tables 1 and 2. Fig. 4 represents the TTT plot of the pig data which shows that the data exhibits an increasing failure rate and Fig. 5 represents the box plots for the data.

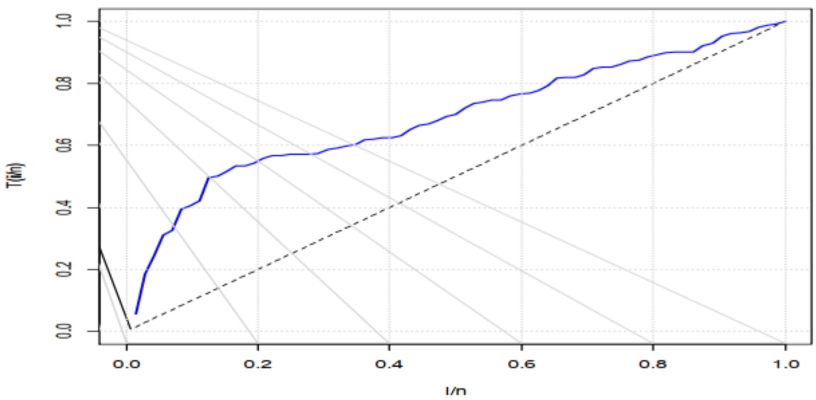

Fig. 4. Graph of Total Time of Test plot (TTT plot) of pig data

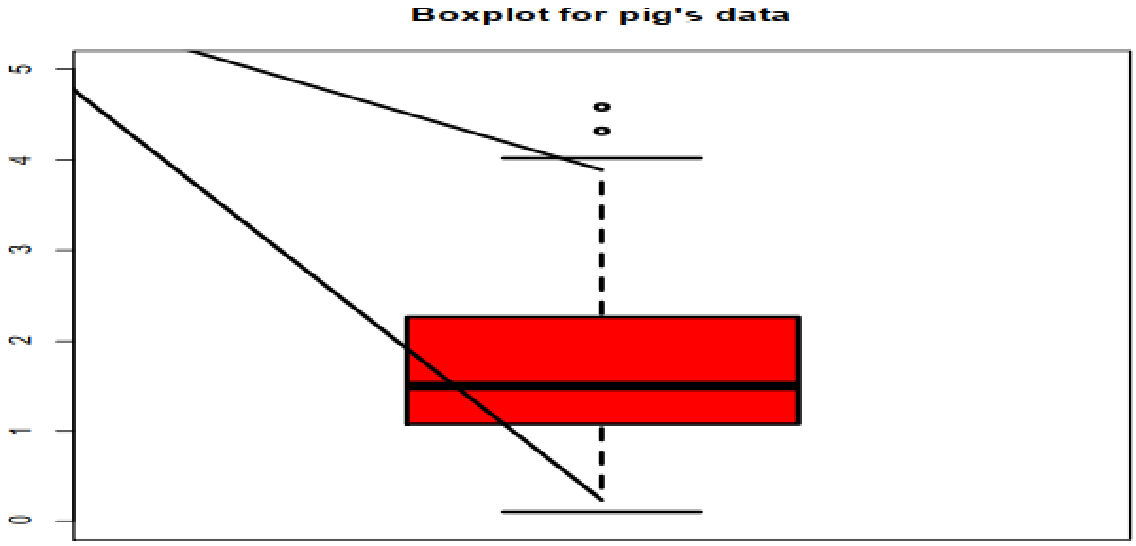

Fig. 5. Boxplot for pig data

Table 1. Data analysis (EDA) of survival time of pigs

min	Q_1	$median$	$mean$	Q_3	max	$kurtosis$	$skew.$	$var.$	$range$
0.10	1.08	1.495	1.768	2.240	5.550	2.225	1.371	1.070	5.45

Table 2. MLEs, standard errors (in parenthesis) and the AIC, BIC and CAIC for Pigs Data

$Model$	ρ	η	l	AIC	BIC	$CAIC$	A^*	W
ER	12.2548 (2.0377)	1.0617 (0.0711)	103.557	211.116	215.669	211.290	0.3659	0.0586
R	0.5655 (0.0471)	$-$ $(-)$	147.431	296.861	299.138	296.919	0.6300	0.1040
E	0.1067 (0.0094)	$-$ $(-)$	113.037	228.074	230.351	228.131	0.4967	0.5977
GT	1.176O (0.0842)	1.176O (0.1328)	118.167	240.334	244.887	240.508	3.3671	0.6001

5 Conclusion

In this work, we have introduced and studied a new model called Extended Rayleigh distribution based on hyperbolic sinh generator. The structural and reliability properties of this distribution have been studied and inferences on parameters have also been examined. The estimation of parameters is carried out by maximum likelihood estimate of the Extended Rayleigh model parameters. The application of the Extended Rayleigh distribution shows that it could provide a better fit than its sub-model.

Competing Interests

Authors have declared that no competing interests exist.

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