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# The Binomial Combinatorial Convolution Sums

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## Abstract

In [1] we can find some formulas of binomial combinatorial convolution sums. Starting from these formulas, we obtain various binomial combinatorial convolution sums.

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## 1 Introduction

Let $\mathbb{N}$ denote the set of positive integers. Further, let $N, d \in \mathbb{N}$ and $s, r, k \in \mathbb{N} \cup \{0\}$. Throughout this paper, we define

$$\sigma_s(N) := \sum_{d|N} d^s, \quad \sigma_{s,r}(N;k) := \sum_{\substack{d|N \\ d \equiv r \bmod k}} d^s, \quad \sigma_s^*(N) := \sum_{\substack{d|N \\ N/d \text{ odd}}} d^s.$$

We note that

$$\sigma_s(N) = \sigma_{s,1}(N;2) + \sigma_{s,0}(N;2) \tag{1.1}$$

and

$$\sigma_{s,1}(2N;2) = \sigma_{s,1}(N;2), \quad \sigma_{s,0}(2N;2) = 2^s \sigma_s(N). \tag{1.2}$$

Moreover we can deduce the property

$$\sigma_s^*(2N) = 2^s \sigma_s^*(N) \quad \text{and} \quad \sigma_s^*(2N) = \sigma_s(2N) - \sigma_s(N). \tag{1.3}$$

For $a, b, n \in \mathbb{N}$, let us define the convolution sum

$$S_{a,b}(n) := \sum_{m=1}^{n-1} \sigma_a(m) \sigma_b(n-m).$$

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Ramanujan showed that the sum $S_{a,b}(n)$ can be evaluated in terms of $\sigma_{a+b+1}(n), \sigma_{a+b-1}(n), ..., \sigma_3(n), \sigma_1(n)$ for the nine pairs $(a, b) \in \mathbb{N}^2$ satisfying $a + b = 2, 4, 6, 8, 12, \quad a \le b, \quad a \equiv b \equiv 1 \pmod 2$. For example, explicitly, we know (see [2]) that

$$S_{1,11}(n) = \frac{691}{65520}\sigma_{13}(n) + \left(\frac{1}{24} - \frac{1}{24}n\right)\sigma_{11}(n) - \frac{691}{65520}\sigma_1(n) \tag{1.4}$$

and (see [3])

$$S_{3,9}(n) = \frac{1}{2640}\sigma_{13}(n) - \frac{1}{240}\sigma_9(n) + \frac{1}{264}\sigma_3(n). \tag{1.5}$$

From [4], we note that for any integer $n \ge 3$, we have

$$\begin{aligned}&\sum_{\substack{(m_1,m_2,m_3)\in\mathbb{N}^3\\ m_1+m_2+m_3=n}} m_1 m_2 \sigma_1(m_1)\sigma_1(m_2)\sigma_1(m_3)\\ &= \frac{1}{288}\left(n^2\sigma_5(n) + (n^2 - 4n^3)\sigma_3(n) - (n^3 - 3n^4)\sigma_1(n)\right).\end{aligned} \tag{1.6}$$

For an elementary proof of (1.4) and (1.5), we refer to [1]. An another interesting arithmetical identity (which was stated by Ramanujan) see [2], for some analytical proofs of this identity, one may refer to [5,6 and 7], also [1] is ; for $n \in \mathbb{N}$, we have

$$\sum_{m=0}^{n-1} \sigma_1(2m+1)\sigma_3(n-m) = \frac{1}{240}\sigma_5(2n+1) - \frac{1}{240}\sigma_3(2n+1). \tag{1.7}$$

We introduce the very important Proposition 1.1, which is the starting point of our paper.

**Proposition 1.1.** *(See ([1], Theorem 12.3)) Let* $k, N \in \mathbb{N}$. *Then*

$$\begin{aligned}&\sum_{s=0}^{k-1}\binom{2k}{2s+1}\left(\sum_{m=1}^{N-1}\sigma_{2k-2s-1}(m)\sigma_{2s+1}(N-m)\right)\\ &= \frac{2k+3}{4k+2}\sigma_{2k+1}(N) + \left(\frac{k}{6} - N\right)\sigma_{2k-1}(N) + \frac{1}{2k+1}\sum_{j=2}^{k}\binom{2k+1}{2j}B_{2j}\sigma_{2k+1-2j}(N).\end{aligned}$$

In Section 2 we obtain some results about the binomial combinatorial convolution sums similar to Proposition 1.1. For instance we have

$$\sum_{s=0}^{k-1} 2^{2s+1}\binom{2k}{2s+1}\sum_{m<\frac{N}{2}}\sigma_{2k-2s-1}(N-2m)\sigma_{2s+1}(m)$$

and

$$\sum_{s=0}^{k-1} 2^{2s+1}\binom{2k}{2s+1}\sum_{m<\frac{N}{2}}\sigma_{2k-2s-1,1}(N-2m;2)\sigma_{2s+1}(m)$$

(see (2.7) and (2.8)).

**Theorem 1.1.** *Let $k, N \in \mathbb{N}$. Then we have*

$$\sum_{s=0}^{k-1} 2^{2s+1} \binom{2k}{2s+1} \sum_{m<N} \sigma_{2k-2s-1,1}(N-m;2)\sigma_{2s+1}(m)$$
$$= \frac{1}{24k+12}\Big[(2k+1)(k-12N)\sigma_{2k-1}(2N) - 4^k(2k+1)(k-6N)\sigma_{2k-1}(N)$$
$$-3\Big\{2^{2(k+1)}(k+1)\sigma_{2k+1}(N) - (2k+3)\sigma_{2k+1}(2N)$$
$$+2^{2k+1}\sum_{j=2}^{k}\binom{2k+1}{2j}B_{2j}\sigma_{2k+1-2j}(N) - 2\sum_{j=2}^{k}\binom{2k+1}{2j}B_{2j}\sigma_{2k+1-2j}(2N)$$
$$+\sum_{j=0}^{2k}\sum_{r=0}^{2k+1-j}(-1)^{j+r}2^jB_j\binom{2k+1}{j}\binom{2k-j+1}{r}\sigma_{2k+1-j-r,1}(N;2)\Big\}\Big].$$

Finally, in Section 3 we consider the convolution sums

$$\sum_{s=0}^{k-1}\binom{2k}{2s+1}\sum_{m<\frac{N}{2}}\sigma_{2k-2s-1}(N-2m)\sigma_{2s+1,1}(m;2)$$

and

$$\sum_{s=0}^{k-1}\binom{2k}{2s+1}\sum_{m<\frac{N}{2}}\sigma_{2k-2s-1}(N-2m)\sigma_{2s+1}(2m)$$

for odd $N$.

# 2 The Binomial Combinatorial Convolution Sums

To prove Lemma 2.1 we need Proposition 2.1.

**Proposition 2.1.** *(See [1]) Let $f : \mathbb{Z} \to \mathbb{C}$ be an even function. Let $N \in \mathbb{N}$. Then we obtain*

$$\sum_{\substack{(a,b,x,y)\in\mathbb{N}^4 \\ ax+by=N \\ a,b \text{ odd}}} (f(a-b) - f(a+b))$$
$$= f(0)\left(\sigma_1^*(N) - \sigma_0^*(N)\right) - \frac{1}{2}\sum_{\substack{d\in\mathbb{N} \\ d|N \\ d \text{ even}}} df(d) - 2\sum_{\substack{d\in\mathbb{N} \\ d|N \\ d \text{ odd}}}\sum_{\substack{t\in\mathbb{N} \\ 1\le t<d \\ t \text{ even}}} f(t).$$

**Lemma 2.1.** *Let $k, N \in \mathbb{N}$. Then we have*

(a)

$$\sum_{s=0}^{k-1}\binom{2k}{2s+1}\sum_{m=1}^{N-1}\sigma_{2k-2s-1,1}(m;2)\sigma_{2s+1,1}(N-m;2) = 2^{2k-1}\sigma_{2k+1}(\frac{N}{2})$$
$$+\frac{1}{4k+2}\sum_{j=0}^{2k}\sum_{r=0}^{2k+1-j}(-1)^{j+r}\binom{2k+1}{j}\binom{2k+1-j}{r}2^jB_j\sigma_{2k+1-j-r,1}(N;2).$$

(b)

$$\sum_{s=0}^{k-1}\binom{2k}{2s+1}\sum_{m=1}^{N-1} m\sigma_{2k-2s-1,1}(m;2)\sigma_{2s+1,1}(N-m;2) = 2^{2(k-1)}N\sigma_{2k+1}(\frac{N}{2})$$
$$+\frac{N}{8k+4}\sum_{j=0}^{2k}\sum_{r=0}^{2k+1-j}(-1)^{j+r}\binom{2k+1}{j}\binom{2k+1-j}{r}2^j B_j\sigma_{2k+1-j-r,1}(N;2).$$

*Proof.* (a) We apply $f(x) = x^{2k}$ in Proposition 2.1. Then the left hand side is

$$\begin{aligned}
\sum_{\substack{(a,b,x,y)\in\mathbb{N}^4\\ ax+by=N\\ a,b \text{ odd}}} (f(a-b)-f(a+b)) &= \sum_{\substack{(a,b,x,y)\in\mathbb{N}^4\\ ax+by=N\\ a,b \text{ odd}}} \left((a-b)^{2k}-(a+b)^{2k}\right)\\
&= \sum_{\substack{(a,b,x,y)\in\mathbb{N}^4\\ ax+by=N\\ a,b \text{ odd}}} \left(\sum_{r=0}^{2k}\binom{2k}{r}(-1)^r a^{2k-r}b^r - \sum_{r=0}^{2k}\binom{2k}{r}a^{2k-r}b^r\right)\\
&= -2\sum_{\substack{(a,b,x,y)\in\mathbb{N}^4\\ ax+by=N\\ a,b \text{ odd}}} \sum_{\substack{r=0\\ r \text{ odd}}}^{2k}\binom{2k}{r}a^{2k-r}b^r \qquad (2.1)\\
&= -2\sum_{s=0}^{k-1}\binom{2k}{2s+1}\sum_{\substack{(a,b,x,y)\in\mathbb{N}^4\\ ax+by=N\\ a,b \text{ odd}}} a^{2k-2s-1}b^{2s+1}.
\end{aligned}$$

Here we can observe that

$$\begin{aligned}
\sum_{\substack{(a,b,x,y)\in\mathbb{N}^4\\ ax+by=N\\ a,b \text{ odd}}} a^{2k-2s-1}b^{2s+1} &= \sum_{m=1}^{N-1}\left(\sum_{\substack{a|m\\ a \text{ odd}}} a^{2k-2s-1}\right)\left(\sum_{\substack{b|N-m\\ b \text{ odd}}} b^{2s+1}\right)\\
&= \sum_{m=1}^{N-1}\sigma_{2k-2s-1,1}(m;2)\sigma_{2s+1,1}(N-m;2).
\end{aligned}$$

So (2.1) becomes

$$-2\sum_{s=0}^{k-1}\binom{2k}{2s+1}\sum_{m=1}^{N-1}\sigma_{2k-2s-1,1}(m;2)\sigma_{2s+1,1}(N-m;2). \qquad (2.2)$$

And the right hand side of Proposition 2.1 is

$$-\frac{1}{2}\sum_{\substack{d|N\\ d \text{ even}}} d^{2k+1} - 2\sum_{\substack{d|N\\ d \text{ odd}}}\sum_{\substack{1\le t<d\\ t \text{ even}}} t^{2k}. \qquad (2.3)$$

Then we have

$$-\frac{1}{2}\sum_{\substack{d|N\\ d \text{ even}}} d^{2k+1} = -\frac{1}{2}\sum_{2l|N}(2l)^{2k+1} = -2^{2k}\sum_{l|\frac{N}{2}} l^{2k+1} = -2^{2k}\sigma_{2k+1}(\frac{N}{2})$$

and

$$\begin{aligned}
&-2\sum_{\substack{d|N\\ d \text{ odd}}}\sum_{\substack{1\le t<d\\ t \text{ even}}} t^{2k} = -2\sum_{\substack{d|N\\ d \text{ odd}}}\sum_{l=1}^{\frac{d-1}{2}}(2l)^{2k} = -2^{2k+1}\sum_{\substack{d|N\\ d \text{ odd}}}\sum_{l=1}^{\frac{d-1}{2}} l^{2k} \\
&= -2^{2k+1}\sum_{\substack{d|N\\ d \text{ odd}}}\frac{1}{2k+1}\sum_{j=0}^{2k}(-1)^j\binom{2k+1}{j}B_j\left(\frac{d-1}{2}\right)^{2k+1-j} \\
&= \sum_{j=0}^{2k}(-1)^{j+1}\binom{2k+1}{j}\frac{2^jB_j}{2k+1}\sum_{\substack{d|N\\ d \text{ odd}}}(d-1)^{2k+1-j} \\
&= \sum_{j=0}^{2k}(-1)^{j+1}\binom{2k+1}{j}\frac{2^jB_j}{2k+1}\sum_{\substack{d|N\\ d \text{ odd}}}\sum_{r=0}^{2k+1-j}\binom{2k+1-j}{r}(-1)^r d^{2k+1-j-r} \\
&= \sum_{j=0}^{2k}(-1)^{j+1}\binom{2k+1}{j}\frac{2^jB_j}{2k+1}\sum_{r=0}^{2k+1-j}\binom{2k+1-j}{r}(-1)^r\sigma_{2k+1-j-r,1}(N;2) \\
&= \sum_{j=0}^{2k}\sum_{r=0}^{2k+1-j}(-1)^{j+r+1}\binom{2k+1}{j}\binom{2k+1-j}{r}\frac{2^jB_j}{2k+1}\sigma_{2k+1-j-r,1}(N;2).
\end{aligned} \tag{2.4}$$

Therefore (2.3) becomes

$$-2^{2k}\sigma_{2k+1}(\frac{N}{2}) + \sum_{j=0}^{2k}\sum_{r=0}^{2k+1-j}(-1)^{j+r+1}\binom{2k+1}{j}\binom{2k+1-j}{r}\frac{2^jB_j}{2k+1}\sigma_{2k+1-j-r,1}(N;2). \tag{2.5}$$

Equating (2.2) and (2.5), we obtain

$$\begin{aligned}
\sum_{s=0}^{k-1}\binom{2k}{2s+1}\sum_{m=1}^{N-1}\sigma_{2k-2s-1,1}(m;2)\sigma_{2s+1,1}(N-m;2) &= 2^{2k-1}\sigma_{2k+1}(\frac{N}{2}) \\
+\frac{1}{4k+2}\sum_{j=0}^{2k}\sum_{r=0}^{2k+1-j}(-1)^{j+r}\binom{2k+1}{j}\binom{2k+1-j}{r}&2^jB_j\sigma_{2k+1-j-r,1}(N;2).
\end{aligned} \tag{2.6}$$

(b) We have

$$\begin{aligned}
&\sum_{s=0}^{k-1}\binom{2k}{2s+1}\sum_{m=1}^{N-1} m\sigma_{2k-2s-1,1}(m;2)\sigma_{2s+1,1}(N-m;2) \\
&= \frac{N}{2}\sum_{s=0}^{k-1}\binom{2k}{2s+1}\sum_{m=1}^{N-1}\sigma_{2k-2s-1,1}(N-m;2)\sigma_{2s+1,1}(m;2),
\end{aligned}$$

since

$$\sum_{m=1}^{N-1} m\sigma_{2k-2s-1,1}(m;2)\sigma_{2s+1,1}(N-m;2)$$
$$= \sum_{m=1}^{N-1} (N-m)\sigma_{2k-2s-1,1}(N-m;2)\sigma_{2s+1,1}(m;2).$$

Therefore, we use Lemma 2.1 (a).

□

**Lemma 2.2.** *Let* $k, N \in \mathbb{N}$. *Then we have*

(a)

$$\sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=1}^{N-1} \sigma_{2k-2s-1,0}(m;2)\sigma_{2s+1,0}(N-m;2)$$
$$= 2^{2k} \left\{ \frac{2k+3}{4k+2}\sigma_{2k+1}(\frac{N}{2}) + \left(\frac{k}{6} - \frac{N}{2}\right) \sigma_{2k-1}(\frac{N}{2}) + \frac{1}{2k+1} \sum_{j=2}^{k} \binom{2k+1}{2j} B_{2j}\sigma_{2k+1-2j}(\frac{N}{2}) \right\}.$$

(b)

$$\sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=1}^{N-1} m\sigma_{2k-2s-1,0}(m;2)\sigma_{2s+1,0}(N-m;2)$$
$$= 2^{2k-1}N \left\{ \frac{2k+3}{4k+2}\sigma_{2k+1}(\frac{N}{2}) + \left(\frac{k}{6} - \frac{N}{2}\right) \sigma_{2k-1}(\frac{N}{2}) + \frac{1}{2k+1} \sum_{j=2}^{k} \binom{2k+1}{2j} B_{2j}\sigma_{2k+1-2j}(\frac{N}{2}) \right\}.$$

*Proof.* (a) From (1.2) we obtain

$$\sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=1}^{N-1} \sigma_{2k-2s-1,0}(m;2)\sigma_{2s+1,0}(N-m;2)$$
$$= \sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=1}^{N-1} 2^{2k-2s-1}\sigma_{2k-2s-1}(\frac{m}{2}) \cdot 2^{2s+1}\sigma_{2s+1}(\frac{N-m}{2})$$
$$= 2^{2k} \sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=1}^{N-1} \sigma_{2k-2s-1}(m)\sigma_{2s+1}(\frac{N}{2}-m).$$

So we use Proposition 1.1.

(b) We obtain

$$\sum_{s=0}^{k-1}\binom{2k}{2s+1}\sum_{m=1}^{N-1}m\sigma_{2k-2s-1,0}(m;2)\sigma_{2s+1,0}(N-m;2)$$
$$=\frac{N}{2}\sum_{s=0}^{k-1}\binom{2k}{2s+1}\sum_{m=1}^{N-1}\sigma_{2k-2s-1,0}(N-m;2)\sigma_{2s+1,0}(m;2),$$

since

$$\sum_{m=1}^{N-1}m\sigma_{2k-2s-1,0}(m;2)\sigma_{2s+1,0}(N-m;2)$$
$$=\sum_{m=1}^{N-1}(N-m)\sigma_{2k-2s-1,0}(N-m;2)\sigma_{2s+1,0}(m;2).$$

Therefore, we refer to Lemma 2.2 (a).

□

**Corollary 2.3.** *Let $k, N \in \mathbb{N}$. Then we have*

(a)

$$\sum_{s=0}^{k-1}\binom{2k}{2s+1}\sum_{m=1}^{N-1}\sigma_{2k-2s-1}(m)\sigma_{2s+1,0}(N-m;2)$$
$$=\frac{1}{24k+12}\Bigg[(2k+1)(k-6N)\sigma_{2k-1}(N)+4^k(2k+1)(k-3N)\sigma_{2k-1}(\frac{N}{2})$$
$$+3\Bigg\{2^{2k+1}\sigma_{2k+1}(\frac{N}{2})+(2k+3)\sigma_{2k+1}(N)$$
$$+2^{2k+1}\sum_{j=2}^{k}\binom{2k+1}{2j}B_{2j}\sigma_{2k+1-2j}(\frac{N}{2})+2\sum_{j=2}^{k}\binom{2k+1}{2j}B_{2j}\sigma_{2k+1-2j}(N)$$
$$-\sum_{j=0}^{2k}\sum_{r=0}^{2k+1-j}(-1)^{j+r}2^jB_j\binom{2k+1}{j}\binom{2k-j+1}{r}\sigma_{2k+1-j-r,1}(N;2)\Bigg\}\Bigg].$$

(b)

$$\sum_{s=0}^{k-1}\binom{2k}{2s+1}\sum_{m=1}^{N-1}\sigma_{2k-2s-1,1}(m;2)\sigma_{2s+1,0}(N-m;2)$$
$$=\sum_{s=0}^{k-1}\binom{2k}{2s+1}\sum_{m=1}^{N-1}\sigma_{2k-2s-1,0}(m;2)\sigma_{2s+1,1}(N-m;2)$$
$$=\frac{1}{24k+12}\Bigg[(2k+1)(k-6N)\sigma_{2k-1}(N)-4^k(2k+1)(k-3N)\sigma_{2k-1}(\frac{N}{2})$$
$$-3\Bigg\{2^{2(k+1)}(k+1)\sigma_{2k+1}(\frac{N}{2})-(2k+3)\sigma_{2k+1}(N)$$
$$+2^{2k+1}\sum_{j=2}^{k}\binom{2k+1}{2j}B_{2j}\sigma_{2k+1-2j}(\frac{N}{2})-2\sum_{j=2}^{k}\binom{2k+1}{2j}B_{2j}\sigma_{2k+1-2j}(N)$$
$$+\sum_{j=0}^{2k}\sum_{r=0}^{2k+1-j}(-1)^{j+r}2^jB_j\binom{2k+1}{j}\binom{2k-j+1}{r}\sigma_{2k+1-j-r,1}(N;2)\Bigg\}\Bigg].$$

*Proof.* (a) From (1.1), we can write

$$\sum_{s=0}^{k-1}\binom{2k}{2s+1}\sum_{m=1}^{N-1}\sigma_{2k-2s-1,1}(m;2)\sigma_{2s+1,1}(N-m;2)$$
$$=\sum_{s=0}^{k-1}\binom{2k}{2s+1}\sum_{m=1}^{N-1}\{\sigma_{2k-2s-1}(m)-\sigma_{2k-2s-1,0}(m;2)\}$$
$$\times\{\sigma_{2s+1}(N-m)-\sigma_{2s+1,0}(N-m;2)\}$$
$$=\sum_{s=0}^{k-1}\binom{2k}{2s+1}\sum_{m=1}^{N-1}\sigma_{2k-2s-1}(m)\sigma_{2s+1}(N-m)$$
$$-2\sum_{s=0}^{k-1}\binom{2k}{2s+1}\sum_{m=1}^{N-1}\sigma_{2k-2s-1}(m)\sigma_{2s+1,0}(N-m;2)$$
$$+\sum_{s=0}^{k-1}\binom{2k}{2s+1}\sum_{m=1}^{N-1}\sigma_{2k-2s-1,0}(m;2)\sigma_{2s+1,0}(N-m;2),$$

since

$$\sum_{s=0}^{k-1}\binom{2k}{2s+1}\sum_{m=1}^{N-1}\sigma_{2k-2s-1}(m)\sigma_{2s+1,0}(N-m;2)$$
$$=\sum_{s=0}^{k-1}\binom{2k}{2s+1}\sum_{m=1}^{N-1}\sigma_{2k-2s-1,0}(m;2)\sigma_{2s+1}(N-m).$$

Thus we use Proposition 1.1, Lemma 2.1 (a), and Lemma 2.2 (a).

(b) Since

$$\sum_{s=0}^{k-1}\binom{2k}{2s+1}\sum_{m=1}^{N-1}\sigma_{2k-2s-1,1}(m;2)\sigma_{2s+1,0}(N-m;2)$$
$$=\sum_{s=0}^{k-1}\binom{2k}{2s+1}\sum_{m=1}^{N-1}\sigma_{2k-2s-1}(m)\sigma_{2s+1,0}(N-m;2)$$
$$-\sum_{s=0}^{k-1}\binom{2k}{2s+1}\sum_{m=1}^{N-1}\sigma_{2k-2s-1,0}(m;2)\sigma_{2s+1,0}(N-m;2),$$

therefore we refer to Lemma 2.2 (a) and Corollary 2.3 (a).

□

*Remark* 2.1. We can rewrite Corollary 2.3 (a) and (b) as follows :

$$\sum_{s=0}^{k-1}\binom{2k}{2s+1}\sum_{m=1}^{N-1}\sigma_{2k-2s-1}(m)\sigma_{2s+1,0}(N-m;2)$$
$$=\sum_{s=0}^{k-1}\binom{2k}{2s+1}\sum_{m=1}^{N-1}\sigma_{2k-2s-1}(m)\cdot 2^{2s+1}\sigma_{2s+1}(\frac{N-m}{2}) \tag{2.7}$$
$$=\sum_{s=0}^{k-1}2^{2s+1}\binom{2k}{2s+1}\sum_{m<\frac{N}{2}}\sigma_{2k-2s-1}(N-2m)\sigma_{2s+1}(m).$$

Similarly,

$$\sum_{s=0}^{k-1}\binom{2k}{2s+1}\sum_{m=1}^{N-1}\sigma_{2k-2s-1,1}(m;2)\sigma_{2s+1,0}(N-m;2)$$
$$=\sum_{s=0}^{k-1}2^{2s+1}\binom{2k}{2s+1}\sum_{m<\frac{N}{2}}\sigma_{2k-2s-1,1}(N-2m;2)\sigma_{2s+1}(m). \tag{2.8}$$

**Corollary 2.4.** *Let $k, N \in \mathbb{N}$. Then we have*

(a)

$$\sum_{s=0}^{k-1}2^{2s+1}\binom{2k}{2s+1}\sum_{m<\frac{N}{2}}\sigma_{2k-2s-1,0}(N-2m;2)\sigma_{2s+1}(m)$$
$$=\frac{2^{2k-1}}{3(2k+1)}\left\{(2k+1)(k-3N)\sigma_{2k-1}(\frac{N}{2})+(6k+9)\sigma_{2k+1}(\frac{N}{2})\right.$$
$$\left.+6\sum_{j=2}^{k}\binom{2k+1}{2j}B_{2j}\sigma_{2k+1-2j}(\frac{N}{2})\right\}.$$

(b)

$$\sum_{s=0}^{k-1} 2^{2s+1}\binom{2k}{2s+1}\sum_{m<\frac{N}{2}}\sigma_{2k-2s-1,0}(N-2m;2)\sigma_{2s+1,0}(m;2)$$
$$=\frac{4^k}{24k+12}\Bigg[(2k+1)(k-3N)\sigma_{2k-1}(\frac{N}{2})+4^k(2k+1)(k-\frac{3}{2}N)\sigma_{2k-1}(\frac{N}{4})$$
$$+3\Bigg\{2^{2k+1}\sigma_{2k+1}(\frac{N}{4})+(2k+3)\sigma_{2k+1}(\frac{N}{2})$$
$$+2^{2k+1}\sum_{j=2}^{k}\binom{2k+1}{2j}B_{2j}\sigma_{2k+1-2j}(\frac{N}{4})+2\sum_{j=2}^{k}\binom{2k+1}{2j}B_{2j}\sigma_{2k+1-2j}(\frac{N}{2})$$
$$-\sum_{j=0}^{2k}\sum_{r=0}^{2k+1-j}(-1)^{j+r}2^jB_j\binom{2k+1}{j}\binom{2k-j+1}{r}\sigma_{2k+1-j-r,1}(\frac{N}{2};2)\Bigg\}\Bigg].$$

(c)

$$\sum_{s=0}^{k-1} 2^{2s+1}\binom{2k}{2s+1}\sum_{m<\frac{N}{2}}\sigma_{2k-2s-1,0}(N-2m;2)\sigma_{2s+1,1}(m;2)$$
$$=\frac{1}{48k+24}\Bigg[16^k\left\{2k(3N-1)+3N-4k^2\right\}\sigma_{2k-1}(\frac{N}{4})$$
$$+2^{2k+1}\Bigg\{(2k+1)(k-3N)\sigma_{2k-1}(\frac{N}{2})-3\cdot 2^{2k+1}\sigma_{2k+1}(\frac{N}{4})+3(2k+3)\sigma_{2k+1}(\frac{N}{2})$$
$$-3\cdot 2^{2k+1}\sum_{j=2}^{k}\binom{2k+1}{2j}B_{2j}\sigma_{2k+1-2j}(\frac{N}{4})+6\sum_{j=2}^{k}\binom{2k+1}{2j}B_{2j}\sigma_{2k+1-2j}(\frac{N}{2})$$
$$+3\sum_{j=0}^{2k}\sum_{r=0}^{2k+1-j}(-1)^{j+r}2^jB_j\binom{2k+1}{j}\binom{2k-j+1}{r}\sigma_{2k+1-j-r,1}(\frac{N}{2};2)\Bigg\}\Bigg].$$

*Proof.* (a) We can see that

$$\sum_{s=0}^{k-1} 2^{2s+1}\binom{2k}{2s+1}\sum_{m<\frac{N}{2}}\sigma_{2k-2s-1,0}(N-2m;2)\sigma_{2s+1}(m)$$
$$=\sum_{s=0}^{k-1} 2^{2s+1}\binom{2k}{2s+1}\sum_{m<\frac{N}{2}}\sigma_{2k-2s-1}(N-2m)\sigma_{2s+1}(m)$$
$$-\sum_{s=0}^{k-1} 2^{2s+1}\binom{2k}{2s+1}\sum_{m<\frac{N}{2}}\sigma_{2k-2s-1,1}(N-2m;2)\sigma_{2s+1}(m).$$

Thus we refer to (2.7) and (2.8).

(b) Since

$$\sum_{s=0}^{k-1} 2^{2s+1}\binom{2k}{2s+1}\sum_{m<\frac{N}{2}}\sigma_{2k-2s-1,0}(N-2m;2)\sigma_{2s+1,0}(m;2)$$
$$= 2^{2k}\sum_{s=0}^{k-1} 2^{2s+1}\binom{2k}{2s+1}\sum_{m<\frac{N}{2}}\sigma_{2k-2s-1}(\frac{N}{2}-m)\sigma_{2s+1}(\frac{m}{2})$$
$$= 2^{2k}\sum_{s=0}^{k-1} 2^{2s+1}\binom{2k}{2s+1}\sum_{m<\frac{N}{4}}\sigma_{2k-2s-1}(\frac{N}{2}-2m)\sigma_{2s+1}(m),$$

therefore we use (2.7).

(c) We note that

$$\sum_{s=0}^{k-1} 2^{2s+1}\binom{2k}{2s+1}\sum_{m<\frac{N}{2}}\sigma_{2k-2s-1,0}(N-2m;2)\sigma_{2s+1,1}(m;2)$$
$$=\sum_{s=0}^{k-1} 2^{2s+1}\binom{2k}{2s+1}\sum_{m<\frac{N}{2}}\sigma_{2k-2s-1,0}(N-2m;2)\sigma_{2s+1}(m)$$
$$-\sum_{s=0}^{k-1} 2^{2s+1}\binom{2k}{2s+1}\sum_{m<\frac{N}{2}}\sigma_{2k-2s-1,0}(N-2m;2)\sigma_{2s+1,0}(m;2).$$

So we apply Corollary 2.4 (a) and (b).

□

***Proof of Theorem 1.1.*** Let $N=2L$ in (2.8). Then, by (1.2), we have

$$\sum_{s=0}^{k-1} 2^{2s+1}\binom{2k}{2s+1}\sum_{m<L}\sigma_{2k-2s-1,1}(2L-2m;2)\sigma_{2s+1}(m)$$
$$=\sum_{s=0}^{k-1} 2^{2s+1}\binom{2k}{2s+1}\sum_{m<L}\sigma_{2k-2s-1,1}(L-m;2)\sigma_{2s+1}(m).$$

So we refer to Corollary 2.3 (b).

□

**Corollary 2.5.** *Let $k, N \in \mathbb{N}$. Then we have*

(a)

$$\sum_{s=0}^{k-1} 2^{2s+1} \binom{2k}{2s+1} \sum_{m=1}^{N-1} \sigma_{2k-2s-1}(2m-1)\sigma_{2s+1}(N-m)$$
$$= \frac{1}{24k+12} \left[ (2k+1)\left(k-6(2N-1)\right)\sigma_{2k-1}(2N-1) \right.$$
$$+ 3\left\{ (2k+3)\sigma_{2k+1}(2N-1) + 2\sum_{j=2}^{k} \binom{2k+1}{2j} B_{2j}\sigma_{2k+1-2j}(2N-1) \right.$$
$$\left.\left. - \sum_{j=0}^{2k}\sum_{r=0}^{2k+1-j} (-1)^{j+r} 2^j B_j \binom{2k+1}{j}\binom{2k-j+1}{r} \sigma_{2k+1-j-r}(2N-1) \right\} \right].$$

(b)

$$\sum_{s=0}^{k-1} 2^{2s+1} \binom{2k}{2s+1} \sum_{m=1}^{N-1} \sigma_{2k-2s-1}(2m)\sigma_{2s+1}(N-m)$$
$$= \frac{1}{24k+12} \left[ (2k+1)(k-12N)\sigma_{2k-1}(2N) + 4^k(2k+1)(k-6N)\sigma_{2k-1}(N) \right.$$
$$+ 3\left\{ 2^{2k+1}\sigma_{2k+1}(N) + (2k+3)\sigma_{2k+1}(2N) \right.$$
$$+ 2^{2k+1}\sum_{j=2}^{k} \binom{2k+1}{2j} B_{2j}\sigma_{2k+1-2j}(N) + 2\sum_{j=2}^{k} \binom{2k+1}{2j} B_{2j}\sigma_{2k+1-2j}(2N)$$
$$\left.\left. - \sum_{j=0}^{2k}\sum_{r=0}^{2k+1-j} (-1)^{j+r} 2^j B_j \binom{2k+1}{j}\binom{2k-j+1}{r} \sigma_{2k+1-j-r,1}(N;2) \right\} \right].$$

*Proof.* (a) Let $N = 2L-1$ in Corollary 2.3 (a). Then we have

$$\sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=1}^{2L-1} \sigma_{2k-2s-1}(m)\sigma_{2s+1,0}(2L-1-m;2)$$
$$= \sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=1}^{L-1} \sigma_{2k-2s-1}(2m-1)\sigma_{2s+1,0}(2L-2m;2)$$
$$= \sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=1}^{L-1} \sigma_{2k-2s-1}(2m-1)\cdot 2^{2s+1}\sigma_{2s+1}(L-m).$$

(b) Let $N = 2L$ in Corollary 2.3 (a). Then we have

$$\sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=1}^{2L-1} \sigma_{2k-2s-1}(m)\sigma_{2s+1,0}(2L-m;2)$$
$$= \sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=1}^{L-1} \sigma_{2k-2s-1}(2m)\sigma_{2s+1,0}(2L-2m;2)$$
$$= \sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=1}^{L-1} \sigma_{2k-2s-1}(2m)\cdot 2^{2s+1}\sigma_{2s+1}(L-m).$$

□

# 3 Convolution Sums for Odd $N$

In this Section we find some convolution sum formulas for odd $N \in \mathbb{N}$.

**Theorem 3.1.** *Let $N$ be a odd positive integer. Then we have*

$$\sum_{s=0}^{k-1}\binom{2k}{2s+1}\sum_{m<\frac{N}{2}}\sigma_{2k-2s-1}(N-2m)\sigma_{2s+1,1}(m;2)$$
$$=\frac{1}{8k+4}\sum_{j=0}^{2k}\sum_{r=0}^{2k+1-j}2^j(-1)^{j+r}B_j\binom{2k+1}{j}\binom{2k+1-j}{r}\sigma_{2k+1-j-r}(N).$$

*Proof.* For odd $N$, we have

$$\sum_{\substack{(a,b,x,y)\in\mathbb{N}^4\\ ax+by=N\\ a,b \text{ odd}}}(f(a-b)-f(a+b))$$
$$=\sum_{\substack{(a,b,x,y)\in\mathbb{N}^4\\ ax+by=N\\ a,b \text{ odd}\\ x,y \text{ odd}}}(f(a-b)-f(a+b))+2\sum_{\substack{(a,b,x,y)\in\mathbb{N}^4\\ ax+by=N\\ a,b \text{ odd}\\ x \text{ odd}\\ y \text{ even}}}(f(a-b)-f(a+b))$$
$$+\sum_{\substack{(a,b,x,y)\in\mathbb{N}^4\\ ax+by=N\\ a,b \text{ odd}\\ x,y \text{ even}}}(f(a-b)-f(a+b))$$
$$=2\sum_{\substack{(a,b,x,y)\in\mathbb{N}^4\\ ax+by=N\\ a,b \text{ odd}\\ x \text{ odd}\\ y \text{ even}}}(f(a-b)-f(a+b)),$$

since there are no elements in $A(N)$ and $B(N)$ where

$$A(N):=\{(a,b,x,y)\in\mathbb{N}^4|ax+by=N \text{ with odd } a,b,x,y, \text{ and odd } N\}$$

and

$$B(N):=\{(a,b,x,y)\in\mathbb{N}^4|ax+by=N \text{ with odd } a,b, \text{ and even } x,y, \text{ and odd } N\}.$$

This concludes that

$$2\sum_{\substack{ax+by=N\\ a,b \text{ odd}\\ x \text{ odd}\\ y \text{ even}\\ N \text{ odd}}}(f(a-b)-f(a+b))=\sum_{\substack{ax+by=N\\ a,b \text{ odd}\\ N \text{ odd}}}(f(a-b)-f(a+b)), \tag{3.1}$$

so we need Proposition 2.1 and let $f(x)=x^{2k}$ in (3.1). As the same manner in (2.1), the left hand side of Eq. (3.1) becomes

$$2\sum_{\substack{ax+by=N\\ a,b \text{ odd}\\ x \text{ odd}\\ y \text{ even}\\ N \text{ odd}}}(f(a-b)-f(a+b)) = -4\sum_{s=0}^{k-1}\binom{2k}{2s+1}\sum_{\substack{ax+by=N\\ a,b \text{ odd}\\ x \text{ odd}\\ y \text{ even}\\ N \text{ odd}}} a^{2k-2s-1}b^{2s+1}$$
$$= -4\sum_{s=0}^{k-1}\binom{2k}{2s+1}\sum_{\substack{m<\frac{N}{2}\\ N \text{ odd}}}\sigma_{2k-2s-1}(N-2m)\sigma_{2s+1,1}(m;2), \tag{3.2}$$

since

$$\sum_{\substack{ax+by=N\\ a,b \text{ odd}\\ x \text{ odd}\\ y \text{ even}\\ N \text{ odd}}} a^{2k-2s-1}b^{2s+1} = \sum_{\substack{ax+2by=N\\ a,b,x \text{ odd}\\ N \text{ odd}}} a^{2k-2s-1}b^{2s+1}$$
$$= \sum_{\substack{m<\frac{N}{2}\\ N \text{ odd}}}\left(\sum_{\substack{a|N-2m\\ a \text{ odd}\\ \frac{N-2m}{a} \text{ odd}}} a^{2k-2s-1}\right)\left(\sum_{\substack{b|m\\ b \text{ odd}}} b^{2s+1}\right)$$
$$= \sum_{\substack{m<\frac{N}{2}\\ N \text{ odd}}}\sigma^{*}_{2k-2s-1,1}(N-2m;2)\sigma_{2s+1,1}(m;2)$$
$$= \sum_{\substack{m<\frac{N}{2}\\ N \text{ odd}}}\left(\sigma_{2k-2s-1,1}(N-2m;2)-\sigma_{2k-2s-1,1}(\frac{N-2m}{2};2)\right)\sigma_{2s+1,1}(m;2)$$
$$= \sum_{\substack{m<\frac{N}{2}\\ N \text{ odd}}}\sigma_{2k-2s-1}(N-2m)\sigma_{2s+1,1}(m;2),$$

where we use (1.3) and $\sigma_{s,1}(\text{odd};2)=\sigma_s(\text{odd})$. And by Proposition 2.1 and (2.4), the right hand side of (3.1) becomes

$$\sum_{\substack{ax+by=N\\ a,b \text{ odd}\\ N \text{ odd}}}(f(a-b)-f(a+b)) = -2\sum_{\substack{d|N\\ d \text{ odd}\\ N \text{ odd}}}\sum_{\substack{1\le t<d\\ t \text{ even}}} t^{2k}$$
$$= \sum_{j=0}^{2k}\sum_{r=0}^{2k+1-j}(-1)^{j+r+1}\binom{2k+1}{j}\binom{2k+1-j}{r}\frac{2^jB_j}{2k+1}\sigma_{2k+1-j-r}(N). \tag{3.3}$$

Equating (3.1) and (3.3), we obtain the proof.

□

**Corollary 3.2.** *Let $N$ be a odd positive integer. Then we have*

$$\sum_{s=0}^{k-1}\binom{2k}{2s+1}\sum_{m<\frac{N}{2}}\sigma_{2k-2s-1}(N-2m)\sigma_{2s+1}(2m)$$
$$=\sum_{s=0}^{k-1}\binom{2k}{2s+1}\sum_{m<\frac{N}{2}}\sigma_{2k-2s-1}(2m-1)\sigma_{2s+1}(N-2m+1)$$
$$=\frac{1}{24k+12}\{(2k+1)(k-6N)\sigma_{2k-1}(N)+3(2k+3)\sigma_{2k+1}(N)$$
$$+6\sum_{j=2}^{k}\binom{2k+1}{2j}B_{2j}\sigma_{2k+1-2j}(N)\}.$$

*Proof.* By (1.1) and (1.2), we can rewrite Theorem 3.1 as

$$\sum_{s=0}^{k-1}\binom{2k}{2s+1}\sum_{m<\frac{N}{2}}\sigma_{2k-2s-1}(N-2m)\sigma_{2s+1,1}(m;2)$$
$$=\sum_{s=0}^{k-1}\binom{2k}{2s+1}\sum_{m<\frac{N}{2}}\sigma_{2k-2s-1}(N-2m)\sigma_{2s+1,1}(2m;2)$$
$$=\sum_{s=0}^{k-1}\binom{2k}{2s+1}\sum_{m<\frac{N}{2}}\sigma_{2k-2s-1}(N-2m)\left\{\sigma_{2s+1}(2m)-\sigma_{2s+1,0}(2m;2)\right\}$$
$$=\sum_{s=0}^{k-1}\binom{2k}{2s+1}\sum_{m<\frac{N}{2}}\sigma_{2k-2s-1}(N-2m)\sigma_{2s+1}(2m)$$
$$+\sum_{s=0}^{k-1}2^{2s+1}\binom{2k}{2s+1}\sum_{m<\frac{N}{2}}\sigma_{2k-2s-1}(N-2m)\sigma_{2s+1}(m).$$

Therefore we use (2.7).

□

# 4 Conclusions

Expanding K. S. Williams' result

$$\sum_{s=0}^{k-1}\binom{2k}{2s+1}\left(\sum_{m=1}^{N-1}\sigma_{2k-2s-1}(m)\sigma_{2s+1}(N-m)\right)$$
$$=\frac{2k+3}{4k+2}\sigma_{2k+1}(N)+\left(\frac{k}{6}-N\right)\sigma_{2k-1}(N)+\frac{1}{2k+1}\sum_{j=2}^{k}\binom{2k+1}{2j}B_{2j}\sigma_{2k+1-2j}(N)$$

(see Proposition 1.1), we obtain some similar formulas for odd and even divisor functions, mainly we show

$$
\begin{aligned}
&\sum_{s=0}^{k-1} 2^{2s+1}\binom{2k}{2s+1}\sum_{m<N}\sigma_{2k-2s-1,1}(N-m;2)\sigma_{2s+1}(m)\\
&=\frac{1}{24k+12}\Big[(2k+1)(k-12N)\sigma_{2k-1}(2N)-4^k(2k+1)(k-6N)\sigma_{2k-1}(N)\\
&\quad-3\Big\{2^{2(k+1)}(k+1)\sigma_{2k+1}(N)-(2k+3)\sigma_{2k+1}(2N)\\
&\quad+2^{2k+1}\sum_{j=2}^{k}\binom{2k+1}{2j}B_{2j}\sigma_{2k+1-2j}(N)-2\sum_{j=2}^{k}\binom{2k+1}{2j}B_{2j}\sigma_{2k+1-2j}(2N)\\
&\quad+\sum_{j=0}^{2k}\sum_{r=0}^{2k+1-j}(-1)^{j+r}2^jB_j\binom{2k+1}{j}\binom{2k-j+1}{r}\sigma_{2k+1-j-r,1}(N;2)\Big\}\Big]
\end{aligned}
$$

in Theorem 1.1.

## Competing Interests

The author declares that no competing interests exist.

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