Journal of Scientific Research & Reports
6(6): 468-475, 2015; Article no.JSRR.2015.171
ISSN: 2320-0227

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Analytical Solution for Normal Depth Problem in a Vertical U-shaped Open Channel Using the Rough Model Method

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Author's contribution

The sole author designed, analyzed and interpreted and prepared the manuscript.

Article Information

DOI: 10.9734/JSRR/2015/16682
Editor(s):
(1) Shangyao Yan, Convener of Civil and Hydraulic Engr Program, NSC, National Central University, Taiwan.
(2) Luigi Rodino, Professor of Mathematical Analysis, Dipartimento di Matematica, Università di Torino, Italy.
Reviewers:
(1) Anonymous, Mexico.
(2) Anonymous, China.
(3) Abdel Razik Ahmed Zidan, Irrigation and Hydraulics Department, Egypt.
Complete Peer review History: http://www.sciencedomain.org/review-history.php?iid=967&id=22&aid=8612

Original Research Article

Received 10th February 2015
Accepted 20th March 2015
Published 28th March 2015

ABSTRACT

Normal depth plays a significant role in the design of open channels and in the analysis of the non-uniform flow as well. Currently there is no analytical method for calculating the normal depth in the open channels. Current methods are either iterative or approximate and consider, unreasonably, Chezy' coefficient or Manning's roughness coefficient as a given data of the problem. Yet, both of these coefficients depend in particular on the normal depth sought and it is therefore unjustified to fix them beforehand. To overcome this drawback, the rough model method (RMM) seems to be the most appropriate tool. The RMM takes into account, in particular, the effect of the absolute roughness which is a readily measurable parameter in practice. The method is based on known referential rough model characteristics used to deduce the normal depth by means of a non-dimensional correction factor.

Keywords: Normal depth; U-shaped channel; rough model method; turbulent flow; discharge; energy slope.

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1. INTRODUCTION

Calculating the normal depth in open channels has been one of the major concerns of the hydraulic engineer. Normal depth plays a significant role in the design of open channels and in the analysis of the non-uniform flow as well. The proposed solutions to this problem were first graphics [1-3], and during recent years the proposed solutions have become approximate or iterative [4-8]. The computation of normal depth in a U-Shaped open channel does no exception to this rule [9]. The focus on this channel is purely practical as it is widely used in hydraulic structures. The most relevant study is certainly that of Swamee and Rathie [10], in which exact analytical equations for normal depth have been reported for rectangular, trapezoidal and circular cross sections. However, the solution is given in terms of an unlimited series whose application to the use of the engineer is not at all handy. The problem in the current methods of calculation is not primarily due to their iterative nature, but stems from the fact that they consider the Manning's roughness coefficient as a given data of the problem, which is undoubtedly unfounded. Manning's roughness coefficient, as well as Chezy's coefficient, is not constant. These two coefficients depend in particular on the normal depth sought. It is therefore unjustified to set beforehand the value of these coefficients as a given data of the problem. The parameter that must be taken into account in the data of the problem is the absolute roughness, which is denoted ε. This is a physical parameter that reflects the state of the inner wall of the channel and which is easily measurable in practice. Currently, there is no explicit or analytical method that considers this parameter. In order to fill this gap and to enrich the literature, this study is proposed. It is based on a new method of calculation known as the rough model method (RMM) which has proven in a recent past [11-18]. This method introduced the absolute roughness ε as a parameter of calculation and does not take into account neither Chezy's coefficient nor Manning's roughness coefficient. This is the particularity that distinguishes it from current methods of calculation. For the calculation of normal depth in U-Shaped open channel, the RMM requires only measurable parameters in practice, namely the discharge Q, the longitudinal slope i, the diameter D of the circular bottom of the channel, the absolute roughness ε and the kinematic viscosity ν of the flowing liquid. The RMM relies on known referential rough model characteristics obtained by application of the Darcy-Weisbach relationship [19]. The friction factor is considered as a constant and well defined in the rough turbulent flow domain. With a non-dimensional correction factor of linear dimension, the characteristics of the rough model are used to derive those of the studied channel, especially normal depth. Note that in the RMM, there is no restriction in the involved parameters and the resulting equations are valid in the entire domain of turbulent flow, corresponding to Reynolds number R **>** 2000 and relative roughness ε/D_h varying in the wide range [0; 0.05]. A calculation example is presented to better understand the calculation procedure and to appreciate its simplicity and efficiency.

2. BASIC EQUATIONS

The rough model method is based on the three well known turbulent flow equations, namely Darcy-Weisbach equation [19], Colebrook-White equation [20] and Reynolds number formula. The Darcy-Weisbach equation gives the longitudinal slope i of the channel as follows:

$$i = \frac{f}{D_h}\frac{Q^2}{2gA^2} \tag{1}$$

where Q is the discharge, g is the acceleration due to gravity, A is the wetted area, D_h is the hydraulic diameter and f is the friction factor given by the now famous Colebrook-White formula as:

$$\frac{1}{\sqrt{f}} = -2\log\left(\frac{\varepsilon / D_h}{3.7} + \frac{2.51}{R\sqrt{f}}\right) \tag{2}$$

where ε is the absolute roughness and R is the Reynolds number which can be expressed as:

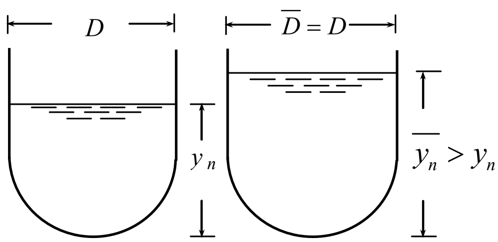
$$R = \frac{4Q}{P\nu} \tag{3}$$

where ν is the kinematic viscosity and P is the wetted perimeter.

3. REFERENTIAL ROUGH MODEL

All geometric and hydraulic characteristics of the rough model are distinguished by the symbol

"$\overline{\ }$". Fig. (1) compares the geometric and hydraulic characteristics of the current channel with those of its rough model. The rough model is particularly characterized by $\overline{\varepsilon} / \overline{D_h} = 0.037$ as the arbitrarily assigned relative roughness value, where $\overline{D_h}$ is the hydraulic diameter. The chosen relative roughness value is so large that the prevailed flow regime is fully rough. Thus, the friction factor is $\overline{f} = 1/16$ according to Eq. (2) for $R = \overline{R}$ tending to infinitely large value. The rough model is also characterized by the width $\overline{D} = D$ and the longitudinal slope $\overline{i} = i$ (see Fig 1). The discharge is $\overline{Q} = Q$ implying that the normal depth $\overline{y_n}$ is such that $\overline{y_n} \neq y_n$ and even $\overline{y_n} > y_n$.



a) b)

Fig. 1. Schematic representation of normal depth in a vertical U-shaped channel. a) current channel. b) rough model

Let us assume for the rough model the aspect ratio $\overline{\eta} = \overline{y_n} / D$, known also as the non-dimensional normal depth. Inasmuch as $\overline{y_n} \neq y_n$, one can write $\overline{\eta} \neq \eta$. Applying Eq. (1) to the rough model leads to:

$$i = \frac{\overline{f}}{\overline{D_h}} \frac{Q^2}{2g\overline{A}^2} \quad (4)$$

Bearing in mind that $\overline{D_h} = 4\overline{A} / \overline{P}$ and $\overline{f} = 1/16$, Eq. (4) can be rewritten as:

$$i = \frac{1}{128g} \frac{\overline{P}}{\overline{A}^3} Q^2 \quad (5)$$

The wetted perimeter $\overline{P}$ and the water area $\overline{A}$ are expressed respectively as:

$$\overline{A} = D^2 (\overline{\eta} - C_0) \quad (6)$$

$$\overline{P} = D(2\overline{\eta} + C_1) \quad (7)$$

where $C_0 = (1 - \pi/4)/2$ and $C_1 = (\pi/2 - 1)$.

Inserting Eq. (6) and Eq. (7) into Eq. (5) and rearranging leads to:

$$Q^{*2} = \frac{(\overline{\eta} - C_0)^3}{2\overline{\eta} + C_1} \quad (8)$$

where Q^* is the relative conductivity expressed as:

$$Q^* = \frac{Q}{8\sqrt{2giD^5}} \quad (9)$$

All the parameters of Eq. (9) are known, which allows determining the value of the relative conductivity Q^*. What is needed is the computation of the aspect ratio $\overline{\eta}$ using Eq. (8) for the given value of Q^*. Let us assume the following change in variables:

$$X = \overline{\eta} - C_0 \quad (10)$$

Thus, Eq. (8) is reduced to:

$$X^3 - 2XQ^{*2} - \frac{\pi}{4} Q^{*2} = 0 \quad (11)$$

Eq. (11) is a cubic equation without second order. Its discriminant can be written as:

$$\Delta = Q^{*4} \left(\frac{\pi}{8} + \frac{2\sqrt{2}}{3\sqrt{3}} Q^* \right) \left(\frac{\pi}{8} - \frac{2\sqrt{2}}{3\sqrt{3}} Q^* \right) \quad (12)$$

Eq. (12) shows that two cases arise:

1. $Q^* \leq \frac{3\pi\sqrt{3}}{16\sqrt{2}}$, then $\Delta \geq 0$. The real root of Eq. (11) is:

$$X = \frac{2\sqrt{2}}{\sqrt{3}} Q^* ch(\beta / 3) \quad (13)$$

where the angle β is as:

$$ch(\beta) = \frac{3\pi\sqrt{3}}{16\sqrt{2}} Q^{*-1} \quad (14)$$

ch is the hyperbolic cosine.

Taking into account Eq. (10), the aspect ratio $\overline{\eta}$ in the rough model is expressed as:

$$\overline{\eta} = C_0 + \frac{2\sqrt{2}}{\sqrt{3}} Q^* ch(\beta / 3) \quad (15)$$

2. $Q^* \geq \frac{3\pi\sqrt{3}}{16\sqrt{2}}$, then $\Delta \leq 0$. The real root of Eq. (11) is:

$$X = \frac{2\sqrt{2}}{\sqrt{3}} Q^* \cos(\beta / 3)$$

where the angle β is as:

$$\cos(\beta) = \frac{3\pi\sqrt{3}}{16\sqrt{2}} Q^{*-1} \quad (16)$$

Taking into account the change in variables given by Eq. (10), the aspect ratio $\overline{\eta}$ in the rough model is then:

$$\overline{\eta} = C_0 + \frac{2\sqrt{2}}{\sqrt{3}} Q^* \cos(\beta / 3) \quad (17)$$

Eq. (15) and Eq. (17) give the exact value of the aspect ratio $\overline{\eta}$ in the rough model.

4. NON-DIMENSIONAL CORRECTION FACTOR OF LINEAR DIMENSION

The rough model method states that any linear dimension *L* of a channel and the linear dimension $\overline{L}$ of its rough model are related by the following equation, applicable to the whole domain of the turbulent flow:

$$L = \psi \overline{L} \quad (18)$$

where ψ is a non-dimensional correction factor of linear dimension, less than unity, which is governed by the following relationship [12,13]:

$$\psi \cong 1.35 \left[-\log\left(\frac{\varepsilon / \overline{D_h}}{4.75} + \frac{8.5}{\overline{R}} \right) \right]^{-2/5} \quad (19)$$

where $\overline{R}$ is the Reynolds number in the rough model given by:

$$\overline{R} = \frac{4Q}{\overline{P}\nu} \quad (20)$$

5. COMPUTATION STEPS OF NORMAL DEPTH

To compute normal depth in a U-Shaped open channel, the following parameters must be given: the discharge *Q*, the diameter *D*, the longitudinal slope *i*, the absolute roughness ε and the kinematic viscosity ν. All these parameters are measurable in practice. The normal depth y_n can be computed according to the following steps:

1. Compute the relative conductivity Q^* using Eq. (9).
2. Determine the aspect ratio $\overline{\eta}$ by the use of Eq. (15) or Eq. (17) in accordance with the sign of the discriminant Δ.
3. Compute the water area $\overline{A}$ and the wetted perimeter $\overline{P}$ using Eq. (6) and Eq. (7) respectively. The hydraulic diameter $\overline{D_h} = 4\overline{A} / \overline{P}$ is then worked out. Use Eq. (20) to compute Reynolds number $\overline{R}$.
4. Knowing $\overline{D_h}$ and $\overline{R}$, compute the non-dimensional correction factor of linear dimension ψ by the use of Eq. (19).
5. Assign to the rough model the following new linear dimension $\overline{D} = D / \psi$ according to the fundamental Eq. (18).
6. Thus, derive the new value of the relative conductivity Q^* using Eq. (9).

7. Applying then one the Eq. (15) or Eq. (17), in accordance with the sign of the discriminant Δ, results in $\overline{\eta} = \eta$.

8. Finally, the required normal depth y_n is then: $y_n = \eta D$

6. PRACTICAL EXAMPLE

Compute the normal depth y_n in the vertical U-Shaped channel shown in Fig. 1 for the following data:

$$Q = 0.843\ m^3/s,\ D = 2.5\ m,\ i = 10^{-5},\ \varepsilon = 10^{-3} m,\ \nu = 10^{-6} m^2/s.$$

1. According to Eq. (9), the relative conductivity Q^* is:

$$Q^* = \frac{Q}{8\sqrt{2giD^5}} = \frac{0.843}{8\times\sqrt{2\times 9.81\times 10^{-5}\times 2.5^5}} = 0.761268859$$

2. According to the calculated value of Q^*, the aspect ratio $\overline{\eta}$ in the rough model is governing by Eq. (17), along with Eq. (16). The angle β is as:

$$\cos(\beta) = \frac{3\pi\sqrt{3}}{16\sqrt{2}} Q^{*-1} = \frac{3\times\pi\times\sqrt{3}}{16\times\sqrt{2}}\times 0.761268859^{-1} = 0.947673442$$

leading to $\beta = 0.324928853\,radian$

According to Eq. (17), the aspect ratio $\overline{\eta}$ in the rough model is then:

$$\overline{\eta} = C_0 + \frac{2\sqrt{2}}{\sqrt{3}} Q^* \cos(\beta/3) = (1-\pi/4)/2 + \frac{2\times\sqrt{2}}{\sqrt{3}}\times 0.761268859\times\cos(0.324928853/3)$$
$$= 1.343163223$$

3. Using Eq. (6) and Eq. (7), the water area $\overline{A}$ and the wetted perimeter $\overline{P}$ are respectively:

$$\overline{A} = D^2(\overline{\eta} - C_0) = 2.5^2\times\left[1.343163223 - (1-\pi/4)/2\right] = 7.724139405 m^2$$

$$\overline{P} = D(2\overline{\eta} + C_1) = 2.5\times\left[2\times 1.343163223 + (\pi/2 - 1)\right] = 8.142806933 m$$

The hydraulic diameter $\overline{D_h} = 4\overline{A}/\overline{P}$ is then:

$$\overline{D_h} = 4\times 7.724139405/8.142806933 = 3.79433749 m$$

Using Eq. (20), Reynolds number $\overline{R}$ is:

$$\overline{R} = \frac{4Q}{\overline{P}\nu} = \frac{4\times 0.843}{8.142806933\times 10^{-6}} = 414107.8166$$

4. According to Eq. (19), the non-dimensional correction factor ψ was easily calculated as:

$$\psi \cong 1.35\left[-\log\left(\frac{\varepsilon / \overline{D_h}}{4.75}+\frac{8.5}{\overline{R}}\right)\right]^{-2/5} = 0.766322709$$

5. Let us assign to the rough model the following new linear dimension:

$$D / \psi = 2.5 / 0.766322709 = 3.26233318m$$

The corresponding value of the relative conductivity Q^* is given by Eq. (9) as:

$$Q^* = \frac{Q}{8\sqrt{2gi(D/\psi)^5}} = \frac{0.843}{8\times\sqrt{2\times 9.81\times 10^{-5}\times 3.26233318^5}} = 0.391351771$$

6. Considering the calculated value of Q^*, the aspect ratio $\overline{\eta}$ in the rough model is governed by Eq. (15) along with Eq. (14). The angle β is as:

$$ch(\beta) = \frac{3\pi\sqrt{3}}{16\sqrt{2}}Q^{*-1} = \frac{3\times\pi\times\sqrt{3}}{16\times\sqrt{2}}\times 0.391351771^{-1} = 1.84344197$$

leading to $\beta = 1.221443165\, radian$

$$\overline{\eta} = \eta = C_0 + \frac{2\sqrt{2}}{\sqrt{3}}Q^*\, ch(\beta/3) = (1-\pi/4)/2 + \frac{2\times\sqrt{2}}{\sqrt{3}}\times 0.391351771\times ch(1.221443165/3) = 0.8$$

7. The require value of normal depth y_n is thus:

$$y_n = \eta D = 0.8\times 2.5 = 2m$$

8. This step aims to verify the validity of the calculations by determining the discharge *Q* using Chezy's equation. The discharge so calculated should be equal to the discharge given in the problem statement. Chezy's equation expresses the discharge *Q* as:

$$Q = CA\sqrt{R_h i}$$

C is the Chezy's coefficient and R_h is the hydraulic radius.

According to the rough model method, the coefficient *C* is related to ψ by the following formula:

$$C = \frac{8\sqrt{2g}}{\psi^{5/2}}$$

Hence:

$$C = \frac{8\sqrt{2g}}{\psi^{5/2}} = \frac{8\times\sqrt{2\times 9.81}}{0.766322709^{5/2}} = 69.93031286 m^{0.5}/s$$

The water area *A* is given by Eq. (6) as:

$$A = D^2(\eta - C_0) = 2.5^2 \times \left[0.8 - (1 - \pi/4)/2\right] = 4.32936926 m^2$$

According to Eq. (7), the wetted perimeter *P* is as:

$$P = D(2\eta + C_1) = 2.5 \times \left[2 \times 0.8 + (\pi/2 - 1)\right] = 5.42699082 m$$

Thus, the hydraulic radius $R_h = A/P$ is:

$$R_h = 4.32936926 / 5.42699082 = 0.79774767 m$$

Thus, according to Chezy, the discharge *Q* is:

$$Q = CA\sqrt{R_h i} \quad = 69.93031286 \times 4.32936926 \times \sqrt{0.79774767 \times 10^{-5}} = 0.84288369 m^3/s \cong 0.843 m^3/s$$

The discharge so calculated and that given in the problem statement are almost equal, which clearly indicates the validity of the calculations.

7. CONCLUSION

The RMM was successfully applied to compute normal depth in a U-Shaped open channel. The method is based on simple hydraulic equations such as Darcy-Weisbach equation, Colebrook-White relationship and Reynolds number formula. The method took into account the effect of absolute roughness and excludes Chezy's coefficient or Manning's roughness coefficient as a given data of the problem. The Darcy-Weisbach relationship was first applied to a referential rough model whose friction factor has been arbitrarily chosen. This led to the establishment of an explicit relation between the aspect ratio and the relative conductivity. The obtained equation was of third degree, which was analytically solved using hyperbolic and trigonometric functions. From the known aspect ratio of the rough model, the non-dimensional normal depth and therefore the normal depth in the studied channel has been deduced, due to a non-dimensional correction factor. The practical example we suggested showed the reliability of the RMM as well as its simplicity and efficiency. The proposed method does not require the coefficients of Chezy and Manning, unlike current methods of calculation. The theoretical development as well as the calculation example we proposed show no restriction in the application of the rough model method. However, it should be applied to other shapes of geometric profiles to observe its scalability and performance. Its application to open channels should also be investigated to solve the problem of the normal depth which remains relevant. Also suggest the application of the rough model method to explicitly compute Chezy and Manning coefficients, whose relations are currently iterative to calculate the normal depth.

COMPETING INTERESTS

Author has declared that no competing interests exist.

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Peer-review history:
The peer review history for this paper can be accessed here:
http://www.sciencedomain.org/review-history.php?iid=967&id=22&aid=8612