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Results of a perturbation theory generating a one-parameter semigroup

Akinola Yussuff Akinyele[1,*], Omotoni Ezekiel Jimoh[1], Jude Babatunde Omosowon[1], Liman Kinbokun Alhassan[1] and Kareem Akanbi Bello[1]

[1] Department of Mathematics, University of Ilorin, Ilorin, Nigeria.
[*] Correspondence: olaakinyele04@gmail.com

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Abstract: This paper consists of the results about ω-order preserving partial contraction mapping using perturbation theory to generate a one-parameter semigroup. We show that adding a bounded linear operator B to an infinitesimal generator A of a semigroup of the linear operator does not destroy A's property. Furthermore, A is the generator of a one-parameter semigroup, and B is a small perturbation so that $A+B$ is also the generator of a one-parameter semigroup.

Keywords: $\omega - OCP_n$; Analytic semigroup; C_0-semigroup; Perturbation.

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1. Introduction

Perturbation theory comprises methods for finding an approximate solution to a problem; in perturbation theory, the solution is expressed as a power series in a small parameter ε. The first term is the known solution to the solvable problem. Successive terms in the series at higher powers of ε usually become smaller. Assume X is a Banach space, $X_n \subseteq X$ is a finite set, $T(t)$ the C_0-semigroup, $\omega - OCP_n$ the ω-order preserving partial contraction mapping, M_m be a matrix, $L(X)$ be a bounded linear operator on X, P_n a partial transformation semigroup, $\rho(A)$ a resolvent set, $\sigma(A)$ a spectrum of A and $A \in \omega - OCP_n$ is a generator of C_0-semigroup. This paper consists of results of ω-order preserving partial contraction mapping generating a one-parameter semigroup.

Akinyele *et al.*, [1] introduced perturbation of the infinitesimal generator in the semigroup of the linear operator. Batty [2] established some spectral conditions for stability of one-parameter semigroup and also in [3] Batty *et al.*, revealed some asymptotic behavior of semigroup of the operator. Balakrishnan [4] obtained an operator calculus for infinitesimal generators of the semigroup. Banach [5] established and introduced the concept of Banach spaces. Chill and Tomilov [6] deduced some resolvent approaches to stability operator semigroup. Davies [7] obtained linear operators and their spectra. Engel and Nagel [8] introduced a one-parameter semigroup for linear evolution equations. Răbiger and Wolf [9] deduced some spectral and asymptotic properties of the dominated operator. Rauf and Akinyele [10] introduced ω-order preserving partial contraction mapping and established its properties, also in [11], Rauf *et al.*, deduced some results of stability and spectra properties on semigroup of a linear operator. Vrabie [12] proved some results of C_0-semigroup and its applications. Yosida [13] established and proved some results on differentiability and representation of one-parameter semigroup of linear operators.

In this paper, we show that adding a bounded linear operator B to an infinitesimal generator A of a semigroup of the linear operator does not destroy A's property. Furthermore, A is the generator of a one-parameter semigroup, and B is a small perturbation so that $A+B$ is also the generator of a one-parameter semigroup.

2. Preliminaries

Definition 1. (C_0-Semigroup) [8] A C_0-Semigroup is a strongly continuous one parameter semigroup of bounded linear operator on Banach space.

Definition 2. (ω-OCP_n)[11] A transformation $\alpha \in P_n$ is called ω-order preserving partial contraction mapping if $\forall x, y \in$ Dom $\alpha : x \leq y \implies \alpha x \leq \alpha y$ and at least one of its transformation must satisfy $\alpha y = y$ such that $T(t+s) = T(t)T(s)$ whenever $t, s > 0$ and otherwise for $T(0) = I$.

Definition 3. (Perturbation) [1] Let $A : D(A) \subseteq X \to X$ be the generator of a strongly continuous semigroup $(T(t))_{t\geq 0}$ and consider a second operator $B : D(B) \subseteq X \to X$ such that the sum $A + B$ generates a strongly continuous semigroup $(S(t))_{t\geq 0}$. We say that A is perturbed by operator B or that B is a perturbation of A.

Definition 4. (Analytic Semigroup) [12] We say that a C_0-semigroup $\{T(t); t \geq 0\}$ is analytic if there exists $0 < \theta \leq \pi$, and a mapping $S : \bar{\mathbb{C}}_\theta \to L(X)$ such that:

1. $T(t) = S(t)$ for each $t \geq 0$;
2. $S(z_1 + z_2) = S(z_1)S(z_2)$ for $z_1, z_2 \in \bar{\mathbb{C}}_\theta$;
3. $\lim_{z_1 \in \bar{\mathbb{C}}_\theta, z_1 \to 0} S(z_1)x = x$ for $x \in X$; and
4. the mapping $z_1 \to S(z_1)$ is analytic from $\bar{\mathbb{C}}_\theta$ to $L(X)$. In addition, for each $0 < \delta < \theta$, the mapping $z_1 \to S(z_1)$ is bounded from $\mathbb{C}_\delta$ to $L(X)$, then the C_0-Semigroup $\{T(t); t \geq 0\}$ is called analytic and uniformly bounded.

Definition 5. (Perturbation class) [7] We say that operator B is a class P perturbation of the generator A of the one-parameter semigroup $T(t)$ if:

$$\begin{cases} A \quad \text{is a closed operator;} \\ \text{Dom}(A) \supseteq \cup_{t\to 0} T(t)(X); \\ \int_0^1 \|BT(t)\| dt < \infty. \end{cases} \tag{1}$$

Note that $BT(t)$ is bounded for all $t > 0$ under conditions $(1)_1$ and $(1)_2$ by the closed graph theorem.

Example 1 (2×2 matrix $M_m(\mathbb{N} \cup \{0\})$). Suppose

$$A = \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}$$

and let $T(t) = e^{tA}$, then

$$e^{tA} = \begin{pmatrix} e^{2t} & e^{I} \\ e^{t} & e^{2t} \end{pmatrix}.$$

Example 2 (3×3 matrix $M_m(\mathbb{N} \cup \{0\})$). Suppose

$$A = \begin{pmatrix} 2 & 2 & 3 \\ 2 & 2 & 2 \\ 1 & 2 & 2 \end{pmatrix}$$

and let $T(t) = e^{tA}$, then

$$e^{tA} = \begin{pmatrix} e^{2t} & e^{2t} & e^{3t} \\ e^{2t} & e^{2t} & e^{2t} \\ e^{t} & e^{2t} & e^{2t} \end{pmatrix}.$$

Example 3 (3×3 matrix $M_m(\mathbb{C})$). Since we have for each $\lambda > 0$ such that $\lambda \in \rho(A)$ where $\rho(A)$ is a resolvent set on X. Suppose we have

$$A = \begin{pmatrix} 2 & 2 & 3 \\ 2 & 2 & 2 \\ 1 & 2 & 2 \end{pmatrix}$$

and let $T(t) = e^{tA_\lambda}$, then

$$e^{tA_\lambda} = \begin{pmatrix} e^{2t\lambda} & e^{2t\lambda} & e^{3t\lambda} \\ e^{2t\lambda} & e^{2t\lambda} & e^{2t\lambda} \\ e^{t\lambda} & e^{2t\lambda} & e^{2t\lambda} \end{pmatrix}.$$

3. Main results

This section present results of one-parameter semigroup generated by ω-OCP_n using perturbation theory.

Theorem 1. *Let $A \in \omega - OCP_n$ be the generator of a one-parameter semigroup $T(t)_{z\geqslant 0}$ on the Banach space X and suppose that*

$$\|T(t)\| \leqslant Me^{at}$$

for all $t \geqslant 0$. If B is a bounded operator on X, then $(A+B)$ is the generator of a one-parameter semigroup $S(t)_{t\geqslant 0}$ on X such that

$$\|S(t)\| \leqslant Me^{(a+M\|B\|)t}$$

for all $t \geqslant 0$ and $B \in \omega - OCP_n$.

Proof. We define the operators $S(t)$ by

$$\begin{aligned} S(t)f :=& T(t)t + \int_{s=0}^{t} T(t-s)BT(s)ds + \int_{s=0}^{t}\int_{u=0}^{s} T(t-s)BT(s-u)BT(u)fduds \\ &+ \int_{s=0}^{t}\int_{u=0}^{s}\int_{v=0}^{u} T(t-s)BT(s-u)BT(u-v)BT(v)fdvduds + \cdots . \end{aligned} \tag{2}$$

The nth term is an n-fold integral whose integrand is a norm continuous function of the variables. It is easy to verify that the series is norm convergent and that

$$\|S(t)f\| \leqslant Me^{at}\|f\| \sum_{n=0}^{\infty} (tM\|B\|)^n/n! = Me^{(a+M\|B\|)t}. \tag{3}$$

for all $f \in X$, $t \geqslant 0$ and $B \in \omega - OCP_n$.

Since $S(s)S(t) = S(s+t)$ and if $f \in X$, then

$$\lim_{t\to 0} \|s(t)f - f\| \leqslant \lim_{t\to 0} \left\{ \|T(t)f - f\| + \sum_{n=1}^{\infty} Me^{at}\|f\|(tM\|B\|)^n/n! \right\} \geqslant 0,$$

so that $s(t)$ is a one-parameter semigroup. If $f \in X$ and $B \in \omega - OCP_n$, then

$$\begin{aligned} &\lim \|t^{-1}(s(t)f - f) - t^{-1}(T(t)f - f) - Bf\| \\ &\leqslant \lim_{t\to 0} t^{-1} \int_0^t T(t-s)BT(s)fds - Bf\| + \lim_{t\to 0} t^{-1}Me^{at}\|f\| \sum_{n=2}^{\infty} (tM\|B\|)^n/n! \geqslant 0. \end{aligned} \tag{4}$$

It follows that f lies in the domain of the generator Y of $S(t)$ if and only if it lies in the domain of A, and that

$$Yf := Af + Bf, \tag{5}$$

for such f.

As well as being illuminating in its own right, (2) easily leads to the identities

$$\begin{aligned} S(t)f &= T(t)f + \int_{s=0}^{t} S(t-s)BT(s)fde \\ &= T(t)f + \int_{s=0}^{t} S(t-s)BT(s)fds \\ &= T(t)f + \int_{s=0}^{t} T(t-s)BS(s)fds. \end{aligned} \tag{6}$$

Hence the proof is complete. □

Theorem 2. *Suppose B is a class P perturbation of the generator A, then*

$$Dom(B) \supseteq Dom(A).$$

If $\varepsilon > 0$ and $A, B \in \omega - OCP_n$, then

$$\|BR(\lambda, A)\| \leqslant \varepsilon\,, \tag{7}$$

for all large enough $\lambda > 0$. Hence B has relative bound 0 with respect to A.

Proof. Combining (1) with the bound

$$\|BT(t)\| \leqslant \|BT(t)\|Me^{a(t-1)}\,,$$

valid for all $t \geqslant 1$, we then see that

$$\int_0^\infty \|BT(t)\|e^{-\lambda t}dt < \infty\,,$$

for all $\lambda > a$. Suppose $\varepsilon > 0$ and $A, B \in \omega - OCP_n$, then for all large enough λ we have

$$\int_0^\infty \|BT(t)\|e^{-\lambda t}dt \leqslant \varepsilon.$$

Now,

$$\int_0^\infty T(t)e^{-\lambda t}fdt = R(\lambda, A)f\,,$$

for all $f \in X$, so by the closedness of B, we see that $R(\lambda, A)f \in Dom(B)$ and

$$\|BR(\lambda, A)f\| \leqslant \varepsilon\|f\|\,,$$

as required to prove (7).

If $g \in Dom(A)$ and we put $f := (\lambda I - A)g$, then we deduce from (7) that

$$\|Bg\| \leqslant \varepsilon\|(\lambda I - A)g\| \leqslant \varepsilon\|Ag\| + \varepsilon\lambda\|g\|\,, \tag{8}$$

for all large enough $\lambda > 0$. This implies the last statement of the theorem and hence the proof is complete. □

Theorem 3. *Assume B is a class P perturbation of the generator A of the one-parameter semigroup $T(t)$ on X, then $B + A$ is the generator of a one-parameter semigroup $S(t)$ on X and $A, B \in \omega - OCP_n$.*

Proof. Let a be small enough that

$$c := \int_0^{2a} \|BT(t)\|dt < 1. \tag{9}$$

We may define $S(t)$ by the convergent series (2) for $0 \leqslant t \leqslant 2a$, and verify as in the proof of Theorem 1 that $S(s)S(t) = S(s+t)$ for all $s, t \geqslant 0$ such that $s + t \leqslant 2a$. We now extend the definition of $S(t)$ inductively for $t \geqslant 2a$ by putting

$$S(t) := (S(a))^n S(t - na)\,, \tag{10}$$

if $n \in \mathbb{N}$ and $na < t \leqslant (n+1)a$. It is straight forward to verify that $S(t)$ is a semigroup. Now suppose that $\|T(t)\| \leqslant N$ for $0 \leqslant t \leqslant a$.

Assume $f \in X$ and $B \in \omega - OCP_n$, then

$$\|S(t)f - f\| \leqslant \|T(t)f - f\| + \sum_{n=1}^{\infty} N\left(\int_0^t \|BS(t)\|ds\right)^n \|f\|,$$

so that

$$\lim_{t\to 0}\|S(t)f - f\| = 0\,, \tag{11}$$

and $S(t)$ is a one-parameter semigroup on X. It is an immediate consequence of the definition that

$$S(t)f = T(t)f + \int_0^t S(t-s)BS(s)fds\,, \tag{12}$$

for all $f \in X$, $B \in \omega - OCP_n$ and all $0 \leqslant t \leqslant a$. Suppose that this holds for all t such that $0 \leqslant t \leqslant na$. If $na \leqslant u \leqslant (n+1)a$, then

$$\begin{aligned} S(u)f &= S(a)S(u-a)f \\ &= S(a)\left\{T(u-a)f + \int_0^{u-a} S(u-a-s)BT(s)fds\right\} \\ &= T(a)T(u-a)f + \int_0^a S(a-s)BT(s)(T(u-a)f)ds + \int_0^{u-a} S(u-s)BT(s)fds \\ &= T(u)f + \int_0^u S(u-s)BT(s)fds. \end{aligned} \tag{13}$$

By induction, (12) holds for all $t \geqslant 0$.

We finally have to identify the generator Y of $S(t)$. The subspace

$$D := \bigcup_{t>0} T(t)\{Dom(A)\},$$

is contained in $Dom(A)$ and is invariant under $T(t)$ and so is a core for A. If $f \in D$, then there exists $g \in Dom(A)$ where $A \in \omega - OCP_n$ and $\varepsilon > 0$ such that $f = T(\varepsilon)g$. Hence,

$$\begin{aligned} \lim_{t\to 0} t^{-1}(S(t)f - f) &= \lim_{t\to 0}(T(t)f - f) + \lim_{t\to 0} t^{-1}\int_0^t T(t-s)(BT(\varepsilon))T(\varepsilon)gds \\ &= Af + (BT(\varepsilon))g \\ &= (A+B)f. \end{aligned} \tag{14}$$

Therefore, $Dom(Y)$ contains D and $Yf(B+A)$ for all $f \in D$ and $A, B \in \omega - OCP_n$. If $f \in Dom(A)$, then there exists a sequence $f_n \in D$ such that $\|f_n - f\| \to 0$ and $\|Af_n - Af\| \to 0$ as $n \to \infty$. It follows by Theorem 2 that $\|Bf_n - Bf\| \to 0$ and hence that Yf_n converges. Since Y is a generator that is closed, then we deduce that

$$Yf = (B+A)f,$$

for all $f \in Dom(A)$ and $A, B \in \omega - OCP_n$. Multiplying (12) by $e^{-\lambda t}$ and integrating over $(0, \infty)$, we see as in the proof of Theorem 2 that if $\lambda > 0$ is large enough, then

$$R(\lambda, Y)f = R(\lambda, A)f + R(\lambda, Y)BR(\lambda, A)f,$$

for all $f \in Y$ and $A, B \in \omega - OCP_n$.

If λ is also large enough that

$$\|BR(\lambda, A)\| < 1,$$

we deduce that

$$R(\lambda, Y) = R(\lambda, A)(I - BR(\lambda, A))^{-1}.$$

Hence,

$$Dom(Y) = Ran(R(\lambda, Y)) = Ran(R(\lambda, A)) = Dom(A),$$

and $Y = A + B$, and this achieve the proof. □

Theorem 4. *Let $A := -H$ where $H = (-\Delta)^n \geqslant 0$ acts in $L^2(\mathbb{R}^N)$. Also let B be a lower order perturbation of the form*

$$(Bf)(x) := \sum_{|\alpha|<2n} a_\alpha(x)(D^\alpha f)(x).$$

If $a_\alpha \in L^{P_\alpha}(\mathbb{R}^N) + L^\infty(\mathbb{R}^N)$ for each α, where $P_\alpha \geqslant 2$ and $P_\alpha > N/(2n - |\alpha|)$, the $A + B$ is the generator of a one-parameter semigroup and B has relative bound 0 with respect to A where $A, B \in \omega - OCP_n$.

Proof. Suppose $A \in \omega - OCP_n$ is the generator of holomorphic semigroup $T(t)$ such that

$$\|T(t)\| \leqslant c_1, \quad \|AT(t) \leqslant c_2/t\|,$$

for all $t \in (0,1)$. And also the operator $B \in \omega - OCP_n$ has domain containing $Dom(A)$ and there exists $\alpha \in (0,1)$, such that

$$\|Bf\| \leqslant \varepsilon\|Af\| + c_3\varepsilon^{-\alpha/(1-\alpha)}\|f\|, \tag{15}$$

for all $f \in Dom(a)$ and $0 < \varepsilon \leqslant 1$. Then

$$\|BT(t)\| \leqslant (c_2 + c_1c_3)t^{-\alpha}, \tag{16}$$

for all $t \in (0,1)$ so that B is a class P perturbation of A and by Theorem 3 under the stated conditions on t and ε, we have

$$\begin{aligned}\|BT(t)f\| &\leqslant \varepsilon\|AT(t)f\| + c_3\varepsilon^{-\alpha/(1-\alpha)}\|T(t)f\| \\ &\leqslant (\varepsilon c_2 t^{-1} + c_1c_3\varepsilon^{-\alpha/(1-\alpha)})\|f\|.\end{aligned}$$

By putting $\varepsilon = t^{1-\alpha}$, then we obtain (16).

Assume $\alpha \in (0,1)$, H is a non-negative self-adjoint operator on P and B is a linear operator with $Dom(B) \geqslant (H)$, we have

$$\|Bf\| \leqslant \varepsilon\|Af\| + c_3\varepsilon^{-\alpha/(1-\alpha)}\|f\|,$$

for all $\varepsilon > 0$ if and only if there is a constant c_4 such that

$$\|Bf\| \leqslant c_4\|Af\|^{\alpha}\|f\|^{1-\alpha},$$

for all $f \in Dom(A)$ and $A, B \in \omega - OCP_n$.

By Theorem 3, it is sufficient to prove that for each α there exists $\beta < 1$ for which

$$X_\alpha := a_\alpha(\cdot)D^\alpha(H+1)^{-\beta}$$

is bounded.

Let $X_\alpha = a_\alpha(Q)b_\alpha(P)$, where

$$b_\alpha(\varepsilon) = \frac{i^{|\alpha|}\varepsilon^\alpha}{(|\varepsilon|^{2n}+1)^\beta}.$$

If $a_\alpha \in L^\infty(\mathbb{R}^N)$, then $\|X\| \leqslant \|a_\alpha\|_\infty\|b_\alpha\|_\infty < \infty$ provided $|\alpha|/2n < \beta < 1$. On the other hand, if $a_\alpha \in L^P(\mathbb{R}^N)$ where $P \geqslant 2$ and $P > N/(2n - |\alpha|)$, then there exists β such that

$$\frac{N + |\alpha|P}{2np} < \beta < 1.$$

This implies that $(|\alpha| - 2n\beta)p + N < 0$ and hence $b_\alpha \in L^p(\mathbb{R}^N)$. $\square$

4. Conclusion

In this paper, it has been established that ω-order preserving partial contraction mapping generates a one-parameter semigroup using a perturbation theory on Banach space by showing that the semigroup of a linear operator is bounded, that B has a relative bound 0 with respect to A, and also that $B + A$ is a generator of the one-parameter semigroup.

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References

[1] Akinyele, A. Y., Saka-Balogun, O. Y., & Adeyemo, O. A. (2019). Perturbation of infinitesimal generator in semigroup of linear operator. *South East Asian Journal of Mathematics and Mathematical Sciences, 15*(3), 53-64.

[2] Batty, C. J. (1996). Spectral conditions for stability of one-parameter semigroups. *journal of Differential Equations, 127*(1), 87-96.

[3] Batty, C. J., Chill, R., & Tomilov, Y. (2002). Strong stability of bounded evolution families and semigroups. *Journal of Functional Analysis, 193*(1), 116-139.

[4] Balakrishnan, A. V. (1959). An operational calculus for infinitesimal generators of semigroups. *Transactions of the American Mathematical Society, 91*(2), 330-353.

[5] Banach, S. (1922). Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales. *Fundamenta Mathematicae, 3*(1), 133-181.

[6] Chill, R., & Tomilov, Y. (2007). Stability Operator Semigroup. *Banach Center Publication 75, Polish Academy of Sciences, Warsaw*, 71-73.

[7] Davies, E. B. (2007). *Linear Operators and their Spectra* (Vol. 106). Cambridge University Press.

[8] Engel, K. J., Nagel, R., & Brendle, S. (2000). *One-Parameter Semigroups for Linear Evolution Equations* (Vol. 194, pp. xxii+-586). New York: Springer.

[9] Räbiger, F., & Wolff, M. P. (2000). Spectral and asymptotic properties of resolvent-dominated operators. *Journal of the Australian Mathematical Society, 68*(2), 181-201.

[10] Rauf, K., & Akinyele, A. Y. (2019). Properties of ω-order-preserving partial contraction mapping and its relation to C_0-semigroup. *International Journal of Mathematics and Computer Science, 14*(1), 61-68.

[11] Rauf, K., Akinyele, A. Y., Etuk, M. O., Zubair, R. O., & Aasa, M. A. (2019). Some result of stability and spectra properties on semigroup of linear operator. *Advances in Pure Mathematics, 9*(01), 43-51.

[12] Vrabie, I. I. (2003). *C_0-semigroup and Application.* Mathematics Studies, 191, Elsevier, North-Holland.

[13] Yosida, K. (1948). On the differentiability and the representation of one-parameter semi-group of linear operators. *Journal of the Mathematical Society of Japan, 1*(1), 15-21.