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Out-of-plane Equilibrium Points in the Photogravitational Restricted Four-body Problem with Oblateness

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Authors' contributions

This work was carried out in collaboration between both authors. Both authors read and approved the final manuscript.

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Abstract

The restricted four-body problem consists of an infinitesimal particle which is moving under the Newtonian gravitational attraction of three massive bodies, called primaries. The three bodies are moving in circles around their common centre of mass fixed at the origin of the coordinate system, according to the solution of Lagrange, where they are always at the vertices of an equilateral triangle. The fourth body does not affect the motion of the primaries. We consider that the primary body P_1 is dominant and is a source of radiation while the other two small primaries P_2 and P_3 modeled as oblate spheroids have equal masses and oblateness coefficients. The out of equilibrium points of the problem are sought and we found that such critical points exist. These points lie in the $xz-$plane in symmetrical positions with respect to $xy-$plane. We investigate numerically the effects of radiation and oblateness on the positions of out-of-plane equilibrium points, their stability, as well as the regions allowed to motion of the infinitesimal body as determined by the zero velocity surface. It is found that radiation and oblateness have strong effects on the positions of the critical points. We examined the stability of these points and found that the out of plane equilibrium points are unstable.

Keywords: Equilibrium points; restricted four-body problem; radiation; oblateness; zero velocity curves.

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1 Introduction

One of the most extensively studied problems in Celestial Mechanics is the Circular restricted three-body problem (CR3BP, for short), described as the motion of an infinitesimal mass moving under the gravitational effects of two finite masses, called primaries, which move in circular orbits around their common center of mass on account of their mutual gravitational attraction and the infinitesimal mass not influencing the motion of the primaries. The restricted four-body problem (R4BP) is perhaps the simplest model after the R3BP and a natural generalization of it. It deals with the motion of an infinitesimal particle under the Newtonian gravitational attraction of three bodies, called primaries, whose trajectories are the solution of the three Newtonian body problems. In the same direction, as several restricted three body problems have given much insight about real three body problem, the R4BP can be used an intermediate step for the exploration of the general, planar or three-dimensional, four-body problem.

Here, we are interested in the special case of the R4BP, the ‘‘Circular equilateral restricted four-body problem (CER4BP)’’, where the primaries have their triangular equilibrium configuration (see Section 2).

In the classical problem, the effects of the gravitational attraction of the infinitesimal body and other perturbations have been ignored. These perturbations include additional forces in the potential of the classical problem which may make it more realistic for certain applications. Perturbations can well arise from the causes such as from the lack of the sphericity, or the triaxiality, oblateness, and radiation forces of the bodies, variation of the masses, the atmospheric drag, the solar wind, Poynting Robertson effect and the action of other bodies.

Unfortunately, the classical circular restricted four-body problem does not consider the case when at least one of the interacting bodies is an intense emitter of radiation. In certain stellar dynamics problems, it is altogether inadequate to consider solely gravitational force. Thus, it is reasonable to modify the classical model by superimposing a radiative repulsion, whose source coincides with the gravitational field of the main bodies. The importance of radiation influence on celestial bodies has been recognized by many scientists, especially in connection with the formation of concentrations of interplanetary and interstellar dust or grains in planetary and binary star systems, as well as the perturbations on artificial satellites (see, e.g., Kalvouridis [1], Bewick et al. [2] and references therein).

The term photogravitational R3BP was introduced by Radzievskii [3]. This extended version of the classical R3BP takes into account only the radiation pressure component of the radiation drag, which is the next most powerful component after the gravitational forces. Later, Radzievskii [4] performed a complete treatment of the behavior of the equilibrium points. Besides five equilibrium points of the classical problem, he found two equilibrium points on the (x, z) plane in symmetrical positions with respect to the (x, y) plane. Many authors (i.e., Perezhogin [5]; Kunitsyn and Perezhogin [6]; Mignard [7]; Simmons et al. [8]; Ragos and Zagouras [9,10]; Singh [11]) developed and extended Radzievskii [3,4] model to introduce more understandable issues related to the motion of the particle in the field of radiating primaries.

On the other hand, the R3BP assumes that the masses concerned are spherically symmetrical in homogeneous layers, but it is found that celestial bodies, such as Saturn and Jupiter, are sufficiently oblate. Therefore, the study of motion of particles under oblateness effect becomes realistic an important field of research. Recently, Douskos and Markellos [12] investigated the out-of-plane equilibrium points in the R3BP with oblateness. They used the three-dimensional equations of motion written by Sharma and SubbaRao [13] and expanded the equations of motion in power series solutions about the oblateness coefficient of the second primary.

It would be very illuminating to recall that, in the general problem of three bodies, there is a particular solution in which the bodies are placed at the vertices of an equilateral triangle, each moving in a Keplerian orbit. This is well known, and was first studied by Lagrange [14]. He found a solution where the three bodies remain at constant distances from each other while they revolved around their common center of mass.

There has been recently an increased interest for this model (Lagrange equilateral triangle configuration) because of its astronautical applications. This model was used, among others, by Pedersen [15], Brumberg [16], Simo [17], Majorana [18], Alvarez-Ramirez and Vidal [19], Baltagiannis and Papadakis [20,21], Ceccaroni and Biggs [22], Burgos-Garcia and Delgado [23], Papadouris and Papadakis [24,25], Kumari and Kushvah [26], Singh and Vincent [27], etc. The model has been used for practical applications by some researchers, among others, Ceccaroni and Biggs [22], Baltagiannis and Papadakis [21], and references therein. However, to the authors' knowledge presently, the equilibrium points out of the orbital plane of the primaries for this model have remained an open problem to date.

The interest in the Lagrange configuration when the primaries receive different perturbing forces, compared to the classical case, has been revived recently. Papadouris and Papadakis [24] investigated the problem when only the first primary body radiates. They studied the existence, location and stability of the equilibrium points, on and out of the orbital plane for two cases depending on the masses of the primaries namely; two equal masses and three equal masses. They observed that existence and stability of the equilibrium points depend on the mass parameters of the primaries and the radiation factor. Besides, closed regions where the motion of the infinitesimal body can be trapped were given. Later, Papadouris and Papadakis [25] studied the simplest symmetric periodic solutions of the problem for the case of two equal masses. They studied the effect of radiation on the distribution of the periodic orbits, their stability, as well as the evolution of the families as the radiation parameter varies. Poincare surface of section of the problem as the dominant primary radiates were illustrated. Using a similar form of their equations of motion, Singh and Vincent [27] studied the out-of-plane equilibrium points of the problem by taking all the primaries as radiation sources with two of the bodies having the same radiation and mass value. They noted that radiation factors have noticeable effects on the locations of the critical points and the zero velocity curves. The stability of these points is found to be unstable.

The stability region of equilibrium points under the oblateness effects of first two bigger primaries was investigated by Kumari and Kushvah [26]. They established eight equilibrium points, two collinear and six non-collinear and observed that the stability regions of the equilibrium points expanded due to the presence of oblateness coefficients and various values of Jacobi constant C. The allowed regions of motion of the infinitesimal body as well as the regions of the basins of attraction for the equilibrium points were given.

In the present paper we deal with the equilibrium points which exist out of the orbital plane in the case when two of the small primaries modeled as oblate spheroids are of equal masses and oblateness coefficients and the dominant primary body is a source of radiation. Also, the infinitesimal body is assumed to have no influence on the motion of these primaries. In this work, we shall use the paper Singh and Vincent [27] as a guide following the same numerical techniques in order to unveil the effect provided by oblateness of the two small primaries and the radiation pressure of the dominant primary on the existence of equilibrium points and their linear stability in the CER4BP. This model could be used to examine the existence of a dust particle in the Sun, Jupiter, Saturn, Spacecraft system.

The paper is organized as follows: Section 2 determines the equations of motion of the considered model-problem. In Section 3, the existence and location of the out-of-plane equilibrium points are investigated while Section 4 is devoted to the surfaces and curves of zero velocity. The regions of allowed motion as determined by the zero velocity surface and corresponding equipotential curves as well as the positions of out of plane points as the radiation and oblateness parameters varies are given. Section 5 establishes their stability; while Section 6 discusses the obtained results and conclusion of the paper.

2 Equations of Motion

Consider three primary bodies P_1, P_2, P_3, called hereafter the primaries, of masses m_1, m_2, m_3, respectively, with $m_1 >> m_2 = m_3$ moving in circles around their center of mass fixed at the origin of the coordinates. These masses always lie at the vertices of equilateral triangle with the dominant body P_1 being on the negative x-axis at the origin of time. A massless particle is moving under the Newtonian gravitational attraction of the primaries and does not affect the motion of the three bodies. The motion of the system is referred to as axes rotating with uniform angular velocity. The mutual distances of the three primaries remain unchanged with respect to time. This configuration is well known to be stable if the masses satisfy the condition of Gascheau's inequality (see for details Gascheau [28] and Baltagiannis and Papadakis [20]). The equations of motion of the problem are derived in a similar way to the classical R3BP (Szebehely, [29]). The system is dimensionless, i.e., the units of measure of length and time are taken so that the sum of the masses and the distance between the primaries is unity, and, also, the Gaussian constant of gravitation G is 1. The equilateral configuration is possible for all distributions of the masses, whilst the fourth body of negligible mass moves in the same plane. The factors characterizing the radiation pressure of the dominant primary P_1 and the oblateness coefficients of the two small primaries (P_2 and P_3) are also taken into account. The notation $q_1 = 1 - \beta_1$ as related to Schuerman [30] is the reduction factor for the mass of the dominant body where β_1 stand for the ratio of the magnitude of radiation (F_r) to gravitational (F_g) force due to the body. It is clear that: If $q_1 = 1$, the radiation pressure has no effect. If $0 < q_1 < 1$, gravitation force exceeds radiation. If $q_1 = 0$, the radiation force balances the gravitational one while for $q_1 < 0$, the gravity is strengthened by radiation, a case which will be considered in our study. The perturbed mean motion n is given by: $n^2 = 1 + \frac{3}{2}(A_2 + A_3)$, where $A_i = \frac{R_E^2 - R_P^2}{5R^2}$, $i = 2,3$ are the oblateness coefficients of oblate bodies P_2 and P_3 respectively with R_E and R_P as the equatorial and polar radii respectively and R is separation between the primaries. In general, we have $0 \le A_i << 1$; A_i is small in the solar system, for details see Sharma and SubbaRao [13].

Let the coordinates of the infinitesimal mass be (x, y), then the coordinates (x_i, y_i) of the primaries are: $x_1 = -\mu\sqrt{3}$, $y_1 = 0$, $x_2 = x_3 = \frac{\sqrt{3}}{2}(1-2\mu)$, $y_2 = -y_3 = -\frac{1}{2}$, relative to the rotating frame of reference *Oxyz*, with $\mu = \frac{m_2}{m_1 + m_2 + m_3} = \frac{m_3}{m_1 + m_2 + m_3}$ is the mass parameter where $\mu \in (0, 1/2)$.

The differential equations of motion in three dimensions in the dimensionless variables and the barycentric–synodic coordinate system are written as (Douskos and Markellos [12]; Papadouris and Papadakis [24]):

$$\begin{aligned} \ddot{x} - 2n\dot{y} &= \Omega_x, \\ \ddot{y} + 2n\dot{x} &= \Omega_y, \\ \ddot{z} &= \Omega_z, \end{aligned} \tag{1}$$

where

$$\Omega(x, y, z) = \frac{n^2}{2}(x^2 + y^2) + \frac{q_1(1-2\mu)}{r_1} + \frac{\mu}{r_2} + \frac{\mu}{r_3} + \frac{\mu A_2}{2r_2^3}\left(1 - \frac{3z^2}{r_2^2}\right) + \frac{\mu A_3}{2r_3^3}\left(1 - \frac{3z^2}{r_3^2}\right), \tag{2}$$

and

$$r_1^2 = (x + \sqrt{3}\mu)^2 + y^2 + z^2,$$

$$r_2^2 = (x - \frac{\sqrt{3}}{2}(1-2\mu))^2 + (y + \frac{1}{2})^2 + z^2,$$

$$r_3^2 = (x - \frac{\sqrt{3}}{2}(1-2\mu))^2 + (y - \frac{1}{2})^2 + z^2.$$

where r_1, r_2 and r_3 are the distances of the infinitesimal body from the primaries, Ω is the photogravitational potential, dots denote time derivatives, the suffixes *x*, *y* and z indicate the partial derivatives of Ω with respect to *x*, *y* and z respectively.

System (1) admits the Jacobian integral

$$V^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2 = 2\Omega(x, y, z) - C \tag{3}$$

where *C* is the Jacobi Integral Constant. The quantity on the right-hand side of (3) should be larger than or equal to zero for every value of time and the given initial conditions. If the velocity V is put equal to zero, the algebraic equation

$$2\Omega(x, y, z) = C \tag{4}$$

will define, by taking various values of the integration constant C (through the initial conditions), a family of surfaces where it is possible for the infinitesimal mass to move on one side and impossible for it to move on the other.

We remark that whenever $q_1 = 1$ and / or $A_2 = A_3 = 0$, the gravitational case of Baltagiannis and Papadakis [20] is recovered. In the case when only the dominant primary body radiates and the other two small primaries are not oblate spheroids ($A_2 = A_3 = 0$), the equations of motion fully coincide with that given by Papadouris and Papadakis [24].

3 Existence and Location of the Out-of-Plane Equilibrium Points

The position of the out-of-plane equilibrium points can be examined solving (1) by setting

$$\dot{x} = \dot{y} = \dot{z} = \ddot{x} = \ddot{y} = \ddot{z} = 0 \tag{5}$$

and considering z≠0. That is, they are the solutions of the equations

$$n^2 x - \frac{q_1(1-2\mu)(x+\sqrt{3}\mu)}{r_1^3} - \frac{\mu(x-\frac{\sqrt{3}}{2}(1-2\mu))}{r_2^3} - \frac{\mu(x-\frac{\sqrt{3}}{2}(1-2\mu))}{r_3^3} - \frac{3\mu A_2(x-\frac{\sqrt{3}}{2}(1-2\mu))(1-\frac{3z^2}{r_2^2})}{2r_2^5}$$
$$- \frac{3\mu A_3(x-\frac{\sqrt{3}}{2}(1-2\mu))(1-\frac{3z^2}{r_3^2})}{2r_3^5} + \frac{3\mu A_2 z^2(x-\frac{\sqrt{3}}{2}(1-2\mu))}{r_2^7} + \frac{3\mu A_3 z^2(x-\frac{\sqrt{3}}{2}(1-2\mu))}{r_3^7} = 0, \tag{6}$$

$$n^2 y-\frac{q_1(1-2\mu)y}{r_1^3}-\frac{\mu(y+\frac{1}{2})}{r_2^3}-\frac{\mu(y-\frac{1}{2})}{r_3^3}-\frac{3\mu A_2(y+\frac{1}{2})(1-\frac{3z^2}{r_2^2})}{2r_2^5}-\frac{3\mu A_3(y-\frac{1}{2})(1-\frac{3z^2}{r_3^2})}{2r_3^5}+\frac{3\mu A_2 z^2(y+\frac{1}{2})}{r_2^7}$$

$$+\frac{3\mu A_3 z^2(y-\frac{1}{2})}{r_3^7}=0\,, \tag{7}$$

$$z\left(\frac{q_1(1-2\mu)}{r_1^3}+\frac{\mu}{r_2^3}+\frac{\mu}{r_3^3}-\frac{3\mu A_2(\frac{z^2}{r_2^4}-\frac{1}{r_2^2})}{2r_2^3}-\frac{3\mu A_3(\frac{z^2}{r_3^4}-\frac{1}{r_3^2})}{2r_3^3}+\frac{3\mu A_2(1-\frac{3z^2}{r_2^2})}{2r_2^5}+\frac{3\mu A_3(1-\frac{3z^2}{r_3^2})}{2r_3^5}\right)=0 \tag{8}$$

With

$$r_1^2=(x+\sqrt{3}\mu)^2+y^2+z^2\,,$$

$$r_2^2=(x-\frac{\sqrt{3}}{2}(1-2\mu))^2+(y+\frac{1}{2})^2+z^2\,,$$

$$r_3^2=(x-\frac{\sqrt{3}}{2}(1-2\mu))^2+(y-\frac{1}{2})^2+z^2$$

If y = 0, (7) is fulfilled (since $A_2=A_3$) and we solve (6) and (8) for y = 0 and z ≠ 0. This results in the following equations

$$n^2 x_0-\frac{q_1(1-2\mu)(x_0+\sqrt{3}\mu)}{r_{10}^3}-\frac{2\mu(x_0-\frac{\sqrt{3}}{2}(1-2\mu))}{r_{20}^3}-\frac{3\mu A_2(x_0-\frac{\sqrt{3}}{2}(1-2\mu))(1-\frac{3z_0^2}{r_{20}^2})}{r_{20}^5}+$$

$$\frac{6\mu A_2 z_0^2(x_0-\frac{\sqrt{3}}{2}(1-2\mu))}{r_{20}^7}=0\,, \tag{9}$$

$$\frac{q_1(1-2\mu)}{r_{10}^3}+\frac{2\mu}{r_{20}^3}-\frac{6\mu A_2(\frac{z_0^2}{r_{20}^4}-\frac{1}{r_{20}^2})}{r_{20}^3}+\frac{3\mu A_2(1-\frac{3z_0^2}{r_{20}^2})}{r_{20}^5}=0\,, \tag{10}$$

Where

$$n^2=1+3A_2\,,$$

$$r_{10}^2=(x_0+\sqrt{3}\mu)^2+z_0^2\,,$$

$$r_{20}^2=r_{30}^2=(x_0-\frac{\sqrt{3}}{2}(1-2\mu))^2+\frac{1}{4}+z_0^2\,,\; A_2=A_3$$

and the subscript '0' is used to denote the equilibrium values.

The positions of the out-of-plane equilibrium points are the solutions of (9) and (10). It is worth mentioning here that out-of-plane equilibrium points do not exist for any combination of the parameters of this model-problem (the mass parameter μ, radiation factor q_1 and oblateness coefficient A_2 of the primaries). An interesting result is that there are combinations of the parameters of the problem for which the critical points may exist.

In the present work, we have considered the two small equal primaries with masses $m_2 = m_3 = \mu = 0.0190$ and a dominant primary body with mass $m_1 = 1 - 2\mu = 0.962$.

We note here that in the previous studies a necessary condition in order to exist critical points out-of-plane is to consider negative values for the radiation factors (see for details Radzievskii [3], Simmons et al. [8], Papadouris and Papadakis [24] and Singh and Vincent [27]). So, following the Radzievskii [3,4] assumptions, we will investigate in this section the existence and location of out-of-plane equilibrium points in the case where $-1 \le q_1 \le 0$ and $0 \le A_2 << 1$.

Now for $\mu = 0.0190$, there are intervals of q_1 of the form $-1 \le q_1 \le 0$ and $0 \le A_2 << 1$ for which there exist two out-of-the plane equilibrium points, which are denoted by L_1^z and L_2^z. Their positions studied via numerical computation using the software package 'Mathematica' are located in the (x , z) plane in symmetrical positions with respect to the (x , y) plane. Tables 1─3 and Figs. 1─3 present the positions of the out-of-plane equilibrium points as A_2 varies, for fixed values of q_1. Evidently, there correspond to cases when we set $q_1 = -0.03, -0.01$ and -0.001 of varying oblateness of the two small primaries. We have observed that as the radiation and oblateness parameters increases the positions of the out-of-plane equilibrium points are affected. As has been already mentioned out-of-plane equilibrium points may exist for other values of mass parameter μ satisfying the condition of Gascheau (inequality) in order to the Lagrange central configuration to be linearly stable.

Table 1. Numerical computations of the out-of-plane equilibrium points for $\mu = 0.0190$, $q_1 = -0.03$ and varying oblateness coefficients

A_2	x_0	$\pm z_0$
0.0000	-0.00244885	2.16893
0.0015	-0.00242942	2.17089
0.0030	-0.00241019	2.17285
0.0045	-0.00239115	2.17482
0.0060	-0.00237231	2.17679
0.0075	-0.00235365	2.17877
0.0090	-0.00233518	2.18075
0.0105	-0.00231689	2.18273
0.0120	-0.00229878	2.18472
0.0135	-0.00228085	2.18672
0.0150	-0.00226310	2.18871
0.0165	-0.00224552	2.19072
0.0180	-0.00222811	2.19272
0.0195	-0.00221087	2.19474

Table 2. Numerical computations of the out-of-plane equilibrium points for $\mu = 0.0190$, $q_1 = -0.01$ and varying oblateness coefficients

A_2	x_0	$\pm z_0$
0.0000	-0.0159853	0.804573
0.0015	-0.0159104	0.803895
0.0030	-0.0158362	0.803216
0.0045	-0.0157627	0.802536
0.0060	-0.0156899	0.801856
0.0075	-0.0156178	0.801174
0.0090	-0.0155463	0.800491
0.0105	-0.0154755	0.799808
0.0120	-0.0154054	0.799123
0.0135	-0.0153359	0.798437
0.0150	-0.0152670	0.797751
0.0165	-0.0151988	0.797063
0.0180	-0.0151312	0.796375
0.0195	-0.0150642	0.795685

Table 3. Numerical computations of the out-of-plane equilibrium points for $\mu = 0.0190$, $q_1 = -0.001$ and varying oblateness coefficients

A_2	x_0	$\pm z_0$
0.0000	-0.0290377	0.306112
0.0015	-0.0289624	0.305499
0.0030	-0.0288880	0.304889
0.0045	-0.0288144	0.304282
0.0060	-0.0287416	0.303680
0.0075	-0.0286695	0.303080
0.0090	-0.0285982	0.302484
0.0105	-0.0285276	0.301892
0.0120	-0.0284578	0.301303
0.0135	-0.0283887	0.300717
0.0150	-0.0283204	0.300134
0.0165	-0.0282527	0.299555
0.0180	-0.0281857	0.298979
0.0195	-0.0281194	0.298406

From Fig. 1, it can be seen that when $\mu = 0.0190, q_1 = -0.03$ and $0 \le A_2 << 1$, the curves (positions of out-of-plane points) increase. The results in Table 1 are the numerical evidence of the critical points. Inspection of Fig. 2 indicates that, when $\mu = 0.0190, q_1 = -0.01$ and $0 \le A_2 << 1$, the curves decrease. The results in Table 2 present the numerical evidence. We observe from Fig. 3 that when $\mu = 0.0190, q_1 = -0.001$ and $0 \le A_2 << 1$, the curves are slightly lower as the critical points move closer to the dominant primary body. A striking illustration of this can be seen in Table 3. As mentioned previously, the dominant primary body is on the negative x-axis at the origin of time. Comparing Figs. 1 and 3 shows that, the positions of out-of- plane equilibrium points moves in opposite direction. Hence, from Figs. 1—3, we see that for $\mu = 0.0190$ and increasing values of radiation and oblateness parameters, the positions of out-of-plane equilibrium points are significantly affected.

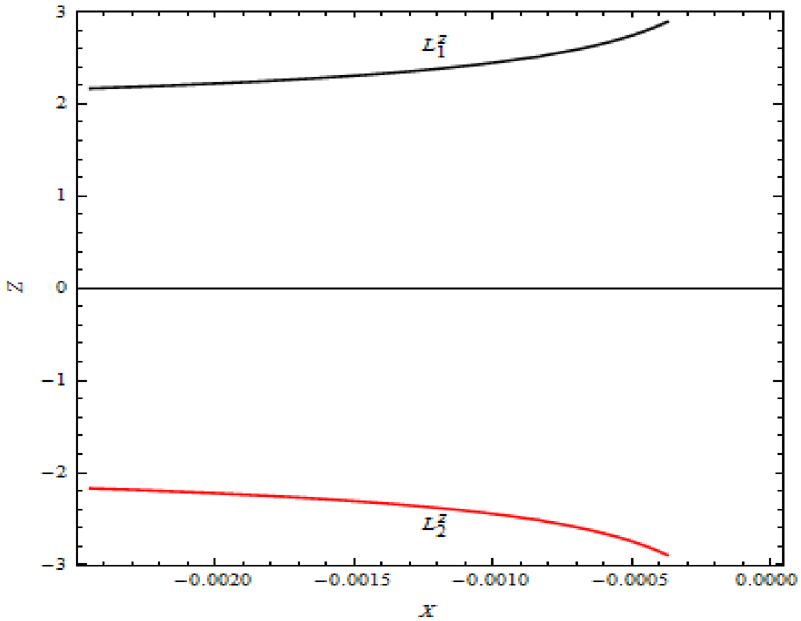

Fig. 1. Position of L_1^z and L_2^z in the (X - Z) plane as a function of A_2 in the interval $0 \le A_2 << 1$, **for** $q_1 = -0.03, \mu = 0.019$

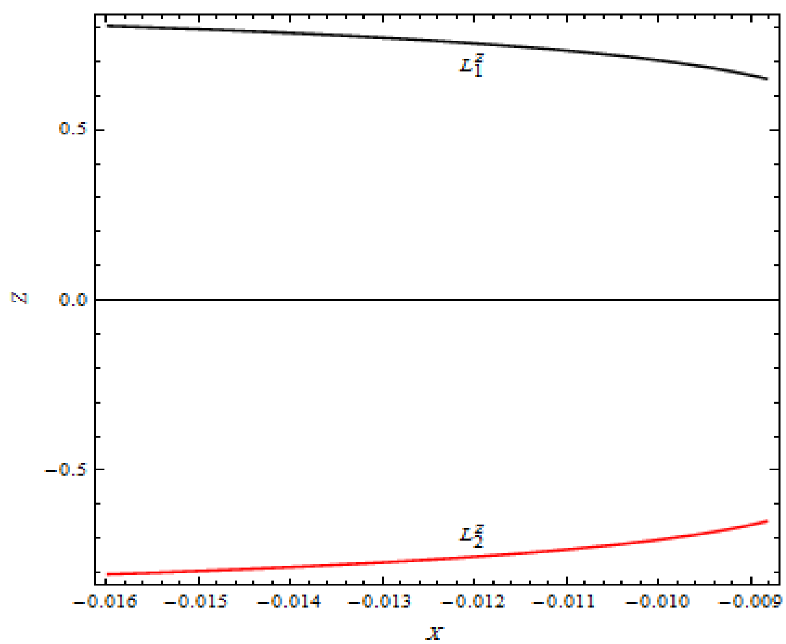

Fig. 2. Position of L_1^z and L_2^z in the (X - Z) plane as a function of A_2 in the interval $0 \le A_2 << 1$, **for** $q_1 = -0.01, \mu = 0.019$

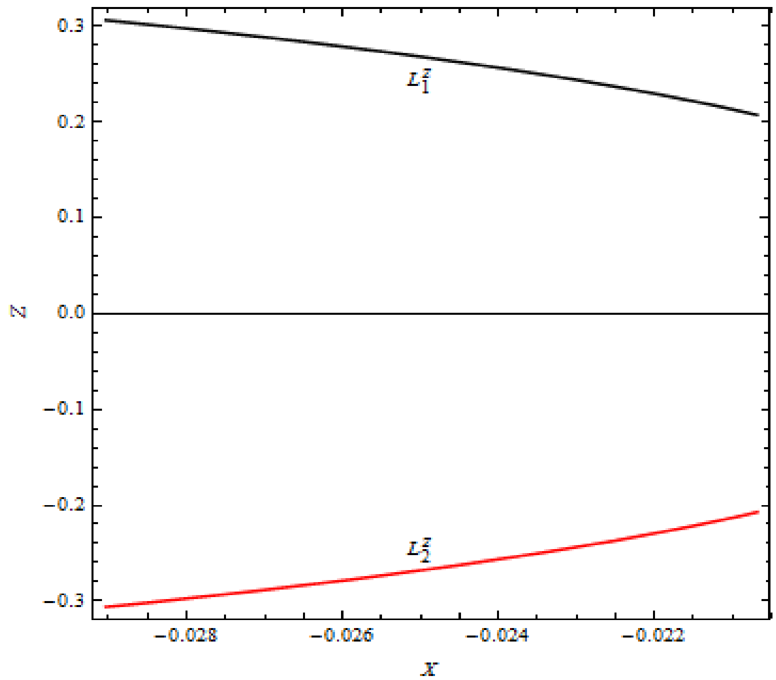

Fig. 3. Position of L_1^z and L_2^z in the (X - Z) plane as a function of A_2 in the interval $0 \le A_2 << 1$, for $q_1 = -0.001, \mu = 0.019$

4 Zero-velocity Curves in the (x, z) Plane

The usefulness of the Jacobi constant integral in clarifying certain general properties of the relative motion of a small body by the construction and investigation of zero velocity curves in every problem of celestial dynamics cannot be overemphasized. For certain initial conditions, these surfaces divide the space into two regions where the infinitesimal fourth body is free to move for various values of the Jacobi constant C. In this section, we present the contours of the surface (3) on the (x , z) plane, for zero velocity, which provide the zero velocity curves. In Fig. 4, we plot these zero velocity curves for $\mu = 0.0190$ and various values of radiation and oblateness parameters i.e., $(q_1 = -0.03, A_2 = 0), (q_1 = -0.01, A_2 = 0.01), (q_1 = -0.001, A_2 = 0.02)$ correspondingly. Large (black) dots indicate the primary bodies, while the small ones are the out- of-plane equilibrium points of the problem. From these figures, it is obvious that radiation and oblateness have significant effects on the structure of the regions allowed to motion of the infinitesimal fourth body. In all cases, between centre of the dominant primary P_1 and its companion out of plane equilibrium points, the zero velocity curves form small ovals of regions not allowed to motion which shrink as the radiation and oblateness parameters varies.

5 Linear Stability of the Out-of-Plane Equilibrium Points

In order to study the linear stability of the out of plane equilibrium points $L_{1,2}^Z$ we transfer the origin to (x_0 ,0, z_0) and linearize the equations of motion, obtaining:

$$\begin{aligned}
&\ddot{\xi} - 2n\dot{\eta} = \xi(\Omega^0_{xx}) + \eta(\Omega^0_{xy}) + \zeta(\Omega^0_{xz}), \\
&\ddot{\eta} + 2n\dot{\xi} = \xi(\Omega^0_{yx}) + \eta(\Omega^0_{yy}) + \zeta(\Omega^0_{yz}), \\
&\ddot{\zeta} = \xi(\Omega^0_{zx}) + \eta(\Omega^0_{zy}) + \zeta(\Omega^0_{zz}),
\end{aligned} \tag{11}$$

where the superscript 'o' indicates that the partial derivatives are to be evaluated at out of plane points (x_0,0, z_0).

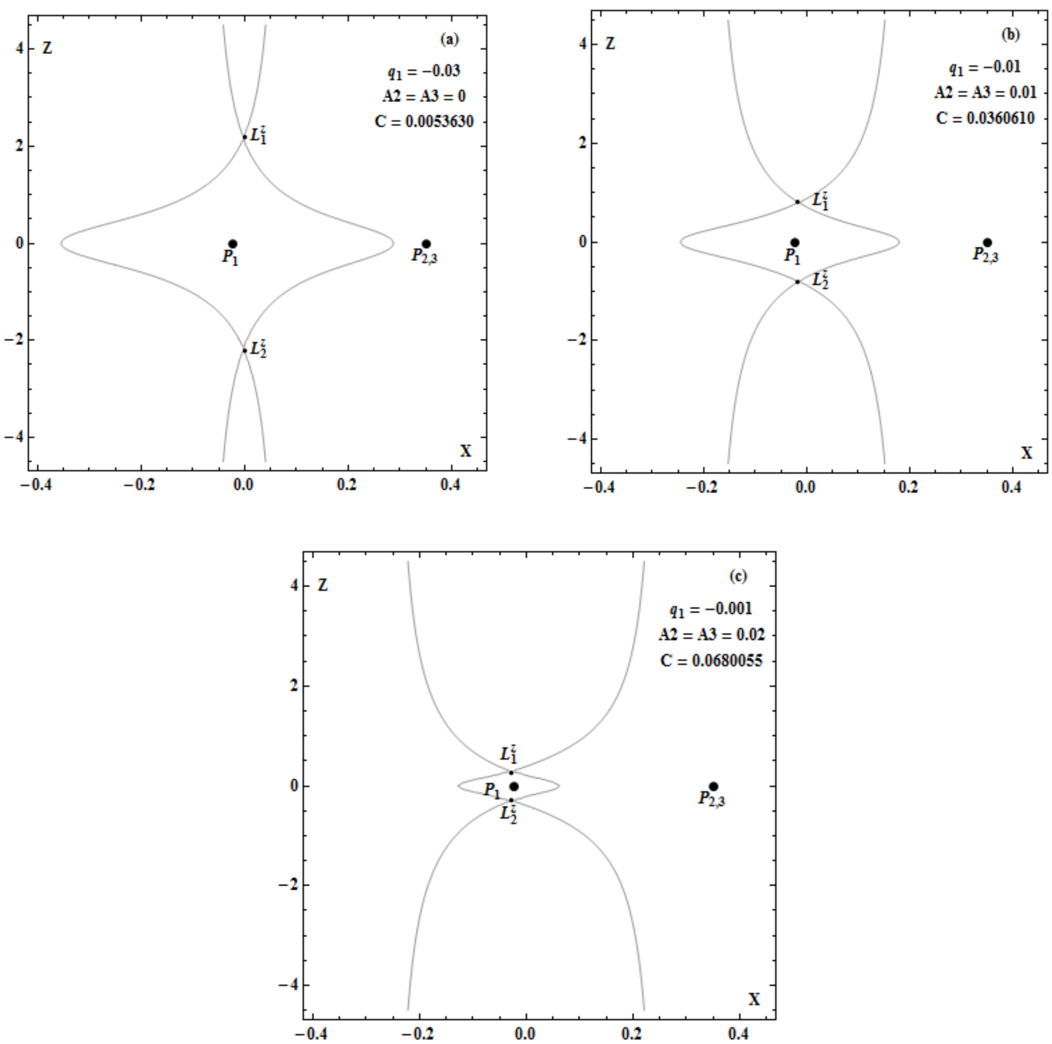

Fig. 4. Zero velocity curves in the (x, z) plane for $\mu = 0.019$ and for $q_1 = -0.03,\ A_{2,3} = 0$ (Frame (a)), $q_1 = -0.01, A_{2,3} = 0.01$ (Frame (b)) and $q_1 = -0.001, A_{2,3} = 0.02$ (Frame (c))

Explicitly, the partial derivatives of system (11) are

$$\Omega^0_{xy} = \Omega^0_{yx} = \Omega^0_{yz} = \Omega^0_{zy} = 0,$$

$$\Omega^0_{xx} = 1 + 3A_2 - \frac{q_1(1-2\mu)}{r_{10}^3}\left[1 - \frac{3(x_0+\sqrt{3}\mu)^2}{r_{10}^2}\right] - \frac{2\mu}{r_{20}^3}\left[1 - \frac{3A_2 z_0^2}{r_{20}^4}\right] - \frac{3A_2\left(r_{20}^2 - 3z_0^2\right)\mu}{r_{20}^7}\left[1 - \frac{5(x_0 - \frac{\sqrt{3}}{2}(1-2\mu))^2}{r_{20}^2}\right]$$

$$+\frac{6(x_0 - \frac{\sqrt{3}}{2}(1-2\mu))^2\mu}{r_{20}^5}\left[1 - \frac{10A_2 z_0^2}{r_{20}^4}\right],$$

$$\Omega^0_{zx} = \Omega^0_{xz} = \frac{3q_1 z_0(1-2\mu)(x_0+\sqrt{3}\mu)}{r_{10}^5} + \frac{6z_0\mu(x_0 - \frac{\sqrt{3}}{2}(1-2\mu))}{r_{20}^5}\left[1 + \frac{2A_2}{r_{20}^2} - \frac{7A_2 z_0^2}{r_{20}^4}\right]$$

$$+3A_2 z_0(x_0 - \frac{\sqrt{3}}{2}(1-2\mu))\mu\left[\frac{11r_{20}^2 - 21z_0^2}{r_{20}^9}\right],$$

$$\Omega^0_{yy} = 1 + 3A_2 - \frac{q_1(1-2\mu)}{r_{10}^3} - \frac{2\mu}{r_{20}^3}\left(1 - \frac{3}{4r_{20}^2}\right) + \frac{6A_2\mu z_0^2}{r_{20}^7}\left(1 - \frac{5}{2r_{20}^2}\right) - \frac{3A_2(r_{20}^2 - 3z_0^2)\mu}{r_{20}^7}\left(1 - \frac{5}{4r_{20}^2}\right),$$

$$\Omega^0_{zz} = -\frac{q_1(1-2\mu)}{r_{10}^3}\left[1 - \frac{3z_0^2}{r_{10}^2}\right] - \frac{2\mu}{r_{20}^3}\left[1 - \frac{3z_0^2}{r_{20}^2}\right] - \frac{3A_2(r_{20}^2 - 3z_0^2)\mu}{r_{20}^7}\left[1 - \frac{5z_0^2}{r_{20}^2}\right] - \frac{6A_2\mu}{r_{20}^9}\left[z_0^2(10z_0^2 - 11r_{20}^2) + r_{20}^4\right]$$

with

$$r_{10}^2 = (x_0 + \sqrt{3}\mu)^2 + z_0^2$$

$$r_{20}^2 = r_{30}^2 = (x_0 - \frac{\sqrt{3}}{2}(1-2\mu))^2 + \frac{1}{4} + z_0^2, \quad A_2 = A_3$$

The characteristic equation corresponding to system (11) is

$$\lambda^6 + a\lambda^4 + b\lambda^2 + c = 0 \tag{12}$$

with

$$a = 4n^2 - \Omega^0 xx - \Omega^0 yy - \Omega^0 zz$$
$$b = \Omega^0 xx\Omega^0 yy + \Omega^0 yy\Omega^0 zz + \Omega^0 zz\Omega^0 xx - 4\Omega^0 zz - (\Omega^0 xz)^2$$
$$c = (\Omega^0 xz)^2\Omega^0 yy - \Omega^0 xx\Omega^0 yy\Omega^0 zz$$

which is a polynomial of sixth degree in λ.

The eigenvalues of the characteristic equation (12) determine the stability or instability of the respective equilibrium points. An equilibrium point will be stable if the characteristic equation (12) has six imaginary roots, otherwise it is unstable. We have computed the characteristic roots $\lambda_i, i = 1,\ldots,6$ as the radiation and oblateness parameters varies with an arbitrary small step and found no case in which all the roots are purely imaginary. Hence, we conclude that the out-of-plane equilibrium points are unstable.

6 Discussion and Conclusion

In this contribution, we study the existence and the stability of out-of-plane equilibrium points in restricted four-body problem formulated on the basis of Lagrangian configuration when the dominant primary is a source of radiation and the other two small primaries modeled as oblate spheroids are of equal masses and oblateness coefficients. Our result shows the existence of two out-of-plane equilibrium points. As it is known, such points do not appear if only the gravitational forces are considered. We observed that as the radiation and oblateness parameters increase, the positions of out-of-plane equilibrium points are significantly affected (see Tables 1—3 and Figs. 1—3). In the absence of oblateness coefficients ($A_2 = A_3 = 0$), our problem correspond to those of Papadouris and Papadakis [24] in the case of two equal masses. In the absence of radiation and oblateness factors ($q_1 = 1, A_2 = A_3 = 0$), our problem correspond to those of Baltagiannis and Papadakis [20]. However, the out-of-plane equilibrium points remains unstable despite the introduction of radiation pressure and oblateness of the dominant and small equal primaries respectively. Radiation and oblateness are seen to have significant effects on the topology of the zero-velocity curves in the (x, z) plane. In particular, between the centre of the dominant primary and its companion out-of-plane equilibrium points, the zero-velocity curves form small ovals of regions not allowed to motion which shrink as the radiation and oblateness parameters vary. Similar phenomenon we observe on the topology of zero velocity curves in the (x, z) plane of a R3BP with oblateness and of photogravitational R4BP by Douskos and Markellos [12] and by Papadouris and Papadakis [24] correspondingly. Most notably, this is the first study, to our knowledge to investigate the existence and stability of out-of-plane equilibrium points in the photogravitational restricted four-body problem with oblateness.

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Competing Interests

Authors have declared that no competing interests exist.

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